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WHEELER-FEYNMAN PROBLEM ON A COMPACT INTERVAL

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Abstract. In this paper the problem (1)+(2) is studied.

1. **Introduction**

In the paper [1] and [3] the author study the Weeler-Feynman problem on R. In this paper we consider the following Weeler-Feynman problem:

$$x'(t) = f(t, x(t), x(t-h), x(t+h)), \quad t \in [a, b], \tag{1}$$

$$x(t) = \varphi(t), \ t \in [t_0 - h, t_0 + h], \tag{2}$$

where $t_0 \in [a, b], \quad a \leq t_0 - h, t_0 + h \leq b$ and $\varphi \in C^1[t_0 - h, t_0 + h]$

2. **Remarks and examples**

2.1. By a solution of (1) we understand a function $x \in C[a-h, b+h] \cap C^1[a, b]$ which satisfies the relation (1) for all $t \in [a, b]$.

2.2. Let $\alpha, \ \beta, \ \gamma \in R, \ \beta \neq 0, \ \gamma \neq 0, \ t_0 \in [a, b]$. We consider the following problem:

$$x'(t) = \alpha x(t) + \beta x(t-h) + \gamma x(t+h)), \quad t \in [a, b], \tag{3}$$

$$x(t) = \varphi(t), \ t \in [t_0 - h, t_0 + h], \tag{4}$$

where $t_0 \in [a, b], \ a \leq [t_0 - h, t_0 + h] \leq b$.

We shall apply the method of steps on intervals $[t_0, b]$ and $[a, t_0]$ to find some "if and only" conditions for the existence of a solution of problem (3)+(4).

Let $t \in [t_0, t_0 + h]$

$$\varphi'(t) = \alpha\varphi(t) + \beta\varphi(t-h) + \gamma x(t+h)$$

Then:

$$x(t) := x_1(t) = \frac{1}{\gamma}[\alpha\varphi(t-h) + \beta\varphi(t-2h) - \varphi'(t-h)],\ t \in [t_0+h, t_0+2h]$$

Let $t \in [t_0+h, t_0+2h]$

$$x_1'(t) = \alpha x_1(t) + \beta\varphi(t-h) + \gamma x(t+h)$$

Then:

$$x(t) := x_2(t) = \frac{1}{\gamma}[\alpha x_1(t-h) + \beta\varphi(t-2h) - x_1'(t-h)],\ t \in [t_0+2h, t_0+3h]$$

By the same way the final step on $[t_0, b]$:

$$x_{n_b}(t) = \frac{1}{\gamma}[\alpha x_{n_b-1}(t-h) + \beta x_{n_b-2}(t-2h) - x'_{n_b-1}(t-h)],\ t \in [t_0+n_b h, b]$$

where $n_b = [\frac{b-t_0}{h}]$.

By the same way on $[a, t_0]$ we find $n_a = [\frac{t_0-a}{h}]$.

Let $n := max\{n_a, n_b\}$.

Let $\varphi \in C^{n+1}[t_0-h, t_0+h]$.

Let $x \in C^n[a-h, b+h] \cap C^{n+1}[a, b]$ be a solution of problem (3)+(4).

We have:

$$x^{(k+1)}(t) = \alpha x^{(k)}(t) + \beta x^{(k)}(t-h) + \gamma x^{(k)}(t+h),\ k \in 0, 1, \ldots, n$$

For $t = t_0$, we have:

$$\varphi^{(k+1)}(t_0) = \alpha\varphi^{(k)}(t_0) + \beta\varphi^{(k)}(t_0-h) + \gamma\varphi^{(k)}(t_0+h),\ k \in \{0, 1, \ldots, n\}$$

Then the problem (3)+(4) has a solution if and only if:

$$\varphi^{(k+1)}(t_0) = \alpha\varphi^{(k)}(t_0) + \beta\varphi^{(k)}(t_0-h) + \gamma\varphi^{(k)}(t_0+h),\ k \in \{0, 1, \ldots, n\}.$$

2.3. For the case in which $\beta = 0$ or $\gamma = 0$ see [2].

3. The main result

In what follow we consider the problem (1)+(2). We need the following conditions.

Let $n_a := [\frac{t_0-a}{h}],\ n_b := [\frac{b-t_0}{h}],\ n := max\{n_a, n_b\}$.

Let $f \in C^{n+1}([a,b] \times R^3)$.

(C1):For all $u_1 \in [a,b],\ u_2, u_4, u_5 \in R$, there exist a unique $u_3 \in R,\ u_3 = f_1(u_1, u_2, u_4, u_5),\ f_1 \in C^{n+1}([a,b] \times R^3)$, such that, $u_5 = f(u_1, u_2, u_3, u_4)$.

(C2):For all $u_1 \in [a,b],\ u_2, u_3, u_5 \in R$, there exist a unique $u_4 \in R,\ u_4 = f_2(u_1, u_2, u_3, u_5),\ f_2 \in C^{n+1}([a,b] \times R^3)$, such that, $u_5 = f(u_1, u_2, u_3, u_4)$.

We have

Theorem 1. *Let $f \in C^{n+1}([a,b] \times R^3)$ satisfies (C1) and (C2). If $\varphi \in C^{n+1}[t_0 - h, t_0 + h]$, then the problem (1)+(2) has a unique solution if and only if φ satisfies the following condition:*

$$\varphi^{(k+1)}(t_0) = [f(t, \varphi(t), \varphi(t-h), \varphi(t+h)]^{(k)}_{t=t_0},\ ,\ k \in \{0, 1, \dots, n\}. \qquad (5)$$

Proof. By the method of steps we construct the solution of (1) +(2) as follows. Let $t \in [t_0, t_0 + h]$

$$\varphi'(t) = f(t, \varphi(t), \varphi(t-h), x(t+h))$$

From (C2) we have

$$x(t) := x_1(t) = f_2(t-h, \varphi(t-h), \varphi(t-2h), \varphi'(t-h)),\ t \in [t_0 + h, t_0 + 2h]\ .$$

By the same method we find the final step:

$$x_{n_b}(t) = f(t-h, x_{n_b-1}(t-h), x_{n_b-1}(t-2h), x\prime_{n_b-1}(t-h)),\ t \in [t_0 + n_b h, b]$$

where $n_b = [\frac{b-t_0}{h}]$.

We must have:

$$\varphi(t_0 + h) = x_1(t_0 + h)$$

$$x_p(t_0 + (p+1)h) = x_{p+1}(t_0 + (p+1)h), \quad p \le n_b - 1$$

By the same way we have the solution on $[a,t_0]$ with the condition

$$\varphi(t_0-h)=x_{-1}(t_0-h)$$

$$x_{-p}(t_0-(p+1)h)=x_{-(p+1)}(t_0-(p+1)h),\quad p\le n_a-1$$

where $n_a=[\frac{t_0-a}{h}]$.

So the solution is:

$$x(t)=\begin{cases} x_{-n_a}(t) & \text{dacă} \quad t\in[a,t_0-n_ah]\\ x_{-k}(t) & \text{dacă} \quad t\in[t_0-(k+1)h,t_0-kh],1\le k\le n_a-1\\ \varphi(t) & \text{dacă} \quad t\in[t_0-h,t_0+h]\\ x_k(t) & \text{dacă} \quad t\in[t_0+kh,t_0+(k+1)h],\quad 1\le k\le n_b-1\\ x_{n_b}(t) & \text{dacă} \quad t\in[t_0+n_bh,b]\end{cases}$$

Let $n=max\{n_a,n_b\}$.

Now we prove the necessity of the condition (5). Let $x\in C[a-h,b+h]\cap C^1[a,b]$ a solution of the problem (1)+(2).

Then $x\in C^n[a-h,b+h]\cap C^{n+1}[a,b]$ is a solution.

We have:

$$x^{(k+1)}(t)=[f(t,x(t),x(t-h),x(t+h))]^{(k)},\ t\in[a,b],\ k\in\{0,1,\dots,n\}.$$

For $t=t_0$, we have (5).

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