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A STUDY OF FUNCTORS ASSOCIATED WITH TOPOLOGICAL GROUPS

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Abstract. The aim of this paper is to construct functors associated with topological groups as well as to investigate these functors. More precisely, we prove that for a given topological groups G there always exists a contravariant functor $F(G)$ from the homotopy category of pointed topological spaces and homotopy classes of base point preserving continuous maps to the category of groups and homomorphisms. We also prove that

(i) the functor $F(G)$ is natural in G in the sense that if the topological groups G and H have the same homotopy type then the groups $F(G)(X)$ and $F(H)(X)$ are isomorphic, for every pointed topological space X; and

(ii) the functor $F(G)$ is homotopy type invariant in the sense that if X and Y are two pointed spaces having the same homotopy type then the groups $F(G)(X)$ are $F(G)(Y)$ are isomorphic.

Moreover, given two topological groups G and H and a continuous homomorphism $\alpha : G \to H$, we show that there always exists a natural transformation between the functors $F(G)$ and $F(H)$ associated with topological groups G and H respectively.

1. **Introduction**

Throughout this paper we assume that (X, x_0) is pointed topological space and maps are base point preserving continuous maps. For simplicity, we write X in place of (X, x_0).

Now we recall following definitions and statements:

Definition 1.1. A pointed topological space is a nonempty topological space with a distinguished element.

Definition 1.2. A pointed topological group is a group G whose underlying set is equipped with a topology such that:

(i) The multiplication map $\mu : G \times G \to G$, given by $(x, y) \mapsto xy$, is continuous if $G \times G$ has the product topology;

(ii) The inversion map $i : G \to G$, given by $x \mapsto x^{-1}$, is continuous.

Then (G, e) is a pointed topological space where e is the identity element.

Definition 1.3. Let $A \subset X$ and let $f_0, f_1 : X \to Y$ be base point preserving continuous maps with $f_0|A = f_1|A$. We write $f_0 \simeq f_1 rel.A$, if there is a continuous map $F : X \times I \to Y$ with $F : f_0 \simeq f_1$ and $F(a, t) = f_0(a) = f_1(a)$, $\forall\ a \in A$ and all $t \in I$. Such a map F is called a homotopy relative to A from f_0 and f_1 and is denoted by $F : f_0 \simeq f_1 rel.A$.

Definition 1.4. If $f : X \to Y$ is base point preserving continuous maps, its homotopy class is the equivalence class $[f] = \{g \in C(X, Y) :\ f \simeq g\}$, where $C(X, Y)$ denotes the set all base point preserving continuous maps from X to Y.

The family of all such homotopy classes is denoted by $[X; Y]$.

Definition 1.5. A base point preserving continuous map $f : X \to Y$ is a homotopy equivalence if there is a base point preserving continuous map $g : Y \to X$ with $g \circ f \simeq I_X$ and $f \circ g \simeq I_Y$. Two spaces X and Y have the same homotopy type denoted by $X \approx Y$ if there is a homotopy equivalence $f : X \to Y$.

Definition 1.6. A category $\mathcal{C}$ consists of

(a) a class of objects $X, Y, Z, \ldots$ denoted by $Ob(\mathcal{C})$;

(b) for each ordered pair of objects X, Y a set of morphisms with domain X and range Y denoted by $\mathcal{C}(X, Y)$;

(c) for each ordered triple of objects X, Y and Z and a pair of morphisms $f : X \to Y$ and $g : Y \to Z$, their composite is denoted by $gf : X \to Z$, satisfying the following two axioms:

(i) *associativity*: if $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Y, Z)$ and $h \in \mathcal{C}(Z, W)$, then $h(gf) = (hg)f \in \mathcal{C}(X, W)$;

(ii) *identity*: for each object Y in $\mathcal{C}$ there is a morphism $I_Y \in \mathcal{C}(Y, Y)$ such that if $f \in \mathcal{C}(X, Y)$, then $I_Y f = f$ and if $h \in \mathcal{C}(Y, Z)$, then $hI_Y = h$.

Definition 1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A contravariant functor T from $\mathcal{C}$ to $\mathcal{D}$ consists of

(i) an object function which assigns to every object X of $\mathcal{C}$ an object $T(X)$ of $\mathcal{D}$; and

(ii) a morphism function which assigns to every morphism $f : X \to Y$ in $\mathcal{C}$, a morphism $T(f) : T(Y) \to T(X)$ in $\mathcal{D}$ such that

(a) $T(I_X) = I_{T(X)}$;

(b) $T(gf) = T(f)T(g)$, for $g : Y \to W$ in $\mathcal{C}$.

Definition 1.8. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Suppose T_1 and T_2 are both contravariant functors from $\mathcal{C}$ and $\mathcal{D}$. A natural transformation ϕ from T_1 to T_2 is a function from the objects of $\mathcal{C}$ to the morphisms of $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$ the following condition hold:

$$\phi(X)T_1(f) = T_2(f)\phi(Y).$$

Lemma 1.9. *Homotopy is an equivalence relation on the set $C(X, Y)$ of all base point preserving continuous maps from X to Y.*

Lemma 1.10. *Let $f_i : X \to Y$ and $g_i : Y \to Z$, for $i = 0, 1$, be continuous. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$; that is $[g_0 \circ f_0] = [g_1 \circ f_1]$.*

In section 2, we construct and investigate functors associated with topological groups.

2. **Functors associated with topological groups**

We now construct functors associated with topological groups.

Let (X, x_0) be a topological space with base point x_0 and (G, e) be a topological group with identity e and $f : X \to G$ be a continuous map such that $f(x_0) = e$.

Now we construct the set M = set of all base preserving continuous maps from (X, x_0) to (G, e).

Then we have the following Proposition:

Proposition 2.1. *Let (X, x_0) be a pointed topological space. The set M of all base point preserving continuous maps from X to G, forms a group under the* *composition '$\Box$' on M defined by*

$$(f_1 \Box f_2)(x) = f_1(x) \cdot f_2(x),\ \forall\ x \in X,\ \ f_1, f_2 \in M,$$

where the right hand side multiplication '$\cdot$' is the multiplication defined on the topological group G.

Proof. First we show that M is nonempty.

Let $C : X \to G$ be defined by $C(x) = e,\ \forall\ x \in X$. Then C is a constant map such that $C \in M \ \Rightarrow\ M \neq \emptyset$.

Let $f_1, f_2 \in M$. Then

$$(f_1 \Box f_2)(x_0) = f_1(x_0) \cdot f_2(x_0) = e \cdot e = e,$$

by definition.

Thus $f_1 \Box f_2$ is a base preserving map. Since G be a topological group and the map, $M \times M \to M$,

$$(f_1, f_2) \mapsto f_1 \Box f_2,\ \forall\ f_1, f_2 \in M,$$

is continuous and hence $f_1 \Box f_2$ is a base point preserving continuous map from X to G. Hence $f_1 \Box f_2 \in M$.

Let $f_1, f_2, f_3 \in M$. Then

$$((f_1 \Box f_2) \Box f_3)(x) = (f_1 \Box f_2)(x) \cdot f_3(x) =$$

$$= (f_1(x) \cdot f_2(x)) \cdot f_3(x) == f_1(x) \cdot (f_2(x) \cdot f_3(x)) =$$

$$= f_1(x) \cdot (f_2 \Box f_3)(x) = (f_1 \Box (f_2 \Box f_3))(x).$$

Thus $((f_1 \Box f_2) \Box f_3)(x) = (f_1 \Box (f_2 \Box f_3))(x),\ \forall\ x \in X$.

Hence $(f_1 \Box f_2) \Box f_3 = f_1 \Box (f_2 \Box f_3)$.

$\Rightarrow$ '$\Box$' associative.

Now

$$(f_1 \Box C)(x) = f_1(x) \cdot C(x) = f_1(x) \cdot e = f_1(x)$$

and

$$(C \Box f_1)(x) = C(x) f_1(x) = e \cdot f_1(x) = f_1(x).$$

Thus $(f_1 \Box C)(x) = (C \Box f_1)(x), \ \forall \ x \in X \ \Rightarrow \ f_1 \Box C = C \Box f_1$.

$\Rightarrow$ C is a identity map from X to G.

Since C is a base point preserving continuous map from X to G and hence $C \in M$.

Let $f_1, f_2 \in M$ such that $(f_1 \Box f_2)(x) = C(x)$

$$\Rightarrow \ f_1(x) \cdot f_2(x) = C(x) \ \Rightarrow \ f_1(x) \cdot f_2(x) = e.$$

Also $f_2(x) \cdot f_1(x) = e$.

Thus $f_1(x) \cdot f_2(x) = f_2(x) \cdot f_1(x) = e$ i.e.

$$(f_1 \Box f_2)(x) = (f_2 \Box f_1)(x) = e, \ \forall \ x \in X.$$

This shows that for each base point preserving continuous map there exists its inverse in M and hence $(M, \Box)$ is a group.

We now carries over the composition '$\Box$' on M to give an operation '$*$' on homotopy classes such that

$$[f] * [g] = [f \Box g], \ \forall \ f, g \in M$$

where $f \Box g$ is defined in Proposition 2.1.

Theorem 2.2. *If X be a pointed topological space and G is a topological group with base point e, then $[X; G]$ is a group.*

Proof. Let X be an arbitrary pointed topological space and G be a topological group.

Let $[X; G]$ = set of all homotopy classes of base point preserving continuous maps from X to G i.e. $[X; G] = \{[f]$ such that $f : X \to G$ is a base point preserving continuous map$\}$.

Now we define a composition '$*$' on $[X; G]$ by the rule:

$$[f] * [g] = [f \Box g], \ \forall \ f, g \in M.$$

$f_1 \in [f]$ and $g_1 \in [g] \ \Rightarrow \ f_1 \simeq f$ and $g_1 \simeq g$ respectively.

$\Rightarrow$ $f_1 \Box g_1 \simeq f \Box g$, as the composite of two homotopic maps are homotopic.

$\Rightarrow$ $[f_1 \Box g_1] = [f \Box g]$, by Lemma 1.10.

$\Rightarrow\ [f_1] * [g_1] = [f] * [g] \ \Rightarrow$ '$*$' is well defined.

Then by using proposition 2.1, $[X; G]$ is a group under the composition '$*$'.

Theorem 2.3. *If $f : X \to Y$ is a base point preserving continuous map, then f induces a homomorphism $f^* : [Y; G] \to [X; G]$, for each topological group G.*

Proof. Define $f^* : [Y; G] \to [X; G]$ by

$$f^*([h]) = [h \circ f], \ \forall\ [h] \in [Y; G].$$

$h_0, h_1 : Y \to G$ and $h_0 \simeq h_1 \ \Rightarrow\ h_0 \circ f \simeq h_1 \circ f \ \Rightarrow\ [h_0 \circ f] = [h_1 \circ f]$, by Lemma 1.10 i.e. $[h_0] = [h_1] \ \Rightarrow\ f^*([h_0]) = f^*([h_1])$. $\Rightarrow$ This map is well defined.

Let $[h_1], [h_2] \in [Y; G]$.

Now $f^*([h_1] * [h_2]) = f^*([h_1 \Box h_2]) = [(h_1 \Box h_2] \circ f]$, by definition. Thus $\forall\ x \in X$,

$$[((h_1 \Box h_2) \circ f)(x)] = [(h_1 \Box h_2)(f(x))] = [h_1(f(x)) \cdot h_2(f(x))],$$

by definition of the product in $[Y; G]$

$$= [(h_1 \circ f)(x) \cdot (h_2 \circ f)(x)] = [((h_1 \circ f) \Box (h_2 \circ f))(x)]$$

$$\Rightarrow\ [(h_1 \Box h_2) \circ f] = [(h_1 \circ f) \Box (h_2 \circ f)] = [h_1 \circ f] * [h_2 \circ f]$$

$$= f^*([h_1] * f^*([h_2]).$$

Thus $f^*([h_1] * [h_2]) = f^*([h_1]) * f^*([h_2]) \ \Rightarrow\ f^*$ is a group homomorphism.

Theorem 2.4. *Let $\alpha : G \to H$ is a continuous group homomorphism between topological groups, then α induces a group homomorphism, $\alpha_* : [X; G] \to [X; H]$.*

Proof. Define $\alpha_* : [X; G] \to [X; H]$ by

$$\alpha_*([f]) = [\alpha \circ f], \ \forall\ f : X \to G.$$

Let $f_1, f_2 : X \to G$ and $f_1 \simeq f_2 \ \Rightarrow\ \alpha \circ f_1 \simeq \alpha \circ f_2$ i.e. $[f_1] = [f_2] \ \Rightarrow$ $[\alpha \circ f_1] = [\alpha \circ f_2] \ \Rightarrow\ \alpha_*([f_1]) = \alpha_*([f_2])$.

Thus this map is well defined.

Let $[f_1], [f_2] \in [X; G]$.

Then $\alpha_*([f_1] * [f_2]) = \alpha_*[(f_1 \Box f_2)] = [\alpha \circ (f_1 \Box f_2)]$, by definition.

Thus $\forall\ x \in X$,

$$[(\alpha \circ (f_1 \Box f_2))(x)] = [\alpha((f_1 \Box f_2)(x))] = [\alpha(f_1(x) \cdot f_2(x))]$$

$$= [\alpha(f_1(x)) \cdot \alpha(f_2(x))] = [(\alpha \circ f_1)(x) \cdot (\alpha \circ f_2)(x)] = [((\alpha \circ f_1) \Box (\alpha \circ f_2))(x)]$$

$$\Rightarrow\ [\alpha \circ (f_1 \Box f_2)] = [(\alpha \circ f_1) \Box (\alpha \circ f_2)] = [\alpha \circ f_1] * [\alpha \circ f_2] = \alpha_*([f_1]) * \alpha_*([f_2]).$$

Thus $\alpha_*([f_1] * [f_2]) = \alpha_*([f_1]) * \alpha_*([f_2])\ \Rightarrow\ \alpha_*$ is a group homomorphism.

Let Htp denote the category of pointed topological spaces and homotopy classes of their base point preserving continuous maps and Grp be the category of groups and their homomorphisms. Then we have the following theorems:

Theorem 2.5. *For a given topological group G, there exists a contravariant functor*

$$F(G) : Htp \to Grp.$$

Proof. Using Theorems 2.2-2.3, define $F(G)(X) = [X; G]$ which is a group and also for $\alpha : X \to Y$ in Htp, $\alpha^* = F(G)(\alpha) : [Y, G] \to [X; G]$ by

$$\alpha^*([g]) = [g \circ \alpha],\ \forall\ [g] \in [Y; G].$$

Let $\alpha : X \to Y$ and $\beta : Y \to Z$ be base point preserving continuous maps, then $\beta \circ \alpha : X \to Z$ is also a base point preserving continuous map.

Thus $(\beta \circ \alpha)^* = F(G)(\beta \circ \alpha) : [Z; G] \to [X; G]$ by

$$(\beta \circ \alpha)^*([g]) = [g \circ (\beta \circ \alpha)],\ \forall\ [g] \in [Z; G].$$

Thus $\forall\ x \in X$,

$$[(g \circ (\beta \circ \alpha))(x)] = [g((\beta \circ \alpha)(x))]$$

$$= [g(\beta(\alpha(x)))] = [(g \circ \beta)(\alpha(x))] = [((g \circ \beta) \circ \alpha)(x)]$$

$$\Rightarrow\ [g \circ (\beta \circ \alpha)] = [(g \circ \beta) \circ \alpha] = \alpha^*([(g \circ \beta)]) = \alpha^*(\beta^*([g])) = (\alpha^* \circ \beta^*)([g]).$$

Thus $\forall\ [g] \in [Z; G]$, $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

Also, for identity map $I_X : X \to X$, $I_X^* = F(G)(I_X) : [X;G] \to [X;G]$ defined by

$$I_X^*([g]) = [g \circ I_X] = [g].$$

Hence $F(G)$ is a contravariant functor.

Given topological groups G and H, $\exists$ two contravariant functors $F(G)$ and $F(H)$. Then $F(G)$ and $F(H)$ have the following relation:

Theorem 2.6. *Given topological groups G, H and a continuous homomorphism $\alpha : G \to H$ there exists a natural transformation*

$$\alpha_* : F(G) \to F(H).$$

Proof. For $[g] \in [Y;G]$ and $f : X \to Y$,

$$F(H)(f)(\alpha_*([g])) = F(H)(f)([\alpha \circ g]) = [(\alpha \circ g) \circ f]$$

i.e. $f^*(\alpha_*([g])) = [(\alpha \circ g) \circ f] \;\Rightarrow\; (f^* \circ \alpha_*)([g]) = [(\alpha \circ g) \circ f]$ and

$$\alpha_*(f^*([g])) = \alpha_*([g \circ f]) = [\alpha \circ (g \circ f)]$$

i.e. $(\alpha_* \circ f^*)([g]) = [\alpha \circ (g \circ f)]$.

Thus $f^* \circ \alpha_* = \alpha_* \circ f^* \;\Rightarrow\; \alpha_*$ is a natural transformation.

Lemma 2.7. *If two topological groups G and H have the same homotopy type, then the homotpy equivalence is a homomorphism.*

Proof. Since G and H have the same homotopy type then there exist continuous maps $f : G \to H$, $g : H \to G$ such that $f(e) = e'$, $g(e') = e$, $g \circ f \simeq I_G$ and $f \circ g \simeq I_H$, where $I_G : G \to G$ and $I_H : H \to H$ are identity maps. Then f and g are both homotopy equivalences.

Since G and H are topological groups, $\exists$ continuous multiplications $\mu : G \times G \to G$ and $\mu' : H \times H \to H$ such that the square

$$\begin{tikzcd} G \times G \arrow[r, "\mu"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu'"'] & H \end{tikzcd}$$

is commutative i.e. $\mu' \circ (f \times f) = f \circ \mu$.

Now $(f \circ \mu)(x, y) = f(\mu(x, y)) = f(xy)$ and

$$(\mu' \circ (f \times f))(x, y) = \mu'((f \times f)(x, y))$$

$$= \mu'(f(x), f(y)) = f(x) \cdot f(y).$$

Thus $f(xy) = f(x) \cdot f(y),\ \forall\ x, y \in G\ \Rightarrow\ f$ is a homomorphism.

Also, g is a homomorphism.

Thus we prove that the homotopy equivalences f and g are continuous group homomorphisms from G to H and H to G respectively.

Theorem 2.8. *If two topological groups G and H are such that G and H have the same homotopy type, then the groups $F(G)(X)$ and $F(H)(X)$ are isomorphic, for every pointed topological space X.*

Proof. Since the topological groups G and H have the same homotopy type, then there exist base point preserving continuous maps $f : G \to H$, $g : H \to G$ such that $g \circ f \simeq I_G$ and $f \circ g \simeq I_H$, where $I_G : G \to G$ and $I_H : H \to H$ are identity maps.

Let $f_* : F(G)(X) \to F(H)(X)$ be defined by

$$f_*([\alpha]) = [f \circ \alpha],\ \forall\ [\alpha] \in F(G)(X).$$

Using Theorem 2.4 and Lemma 2.7, f_* is a homomorphism from $F(G)(X)$ to $F(H)(X)$.

Then f_* satisfies the following properties:

(i) if $f \simeq g\ \Rightarrow\ f_* = g_*$;

(ii) $I_G : G \to G\ \Rightarrow\ I_{G*} = Id_{F(G)(X)}$;

(iii) $(g \circ f)_* = g_* \circ f_*$

for $(g \circ f)_* : F(G)(X) \to F(G)(X)$ defined by

$$(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha], \ \forall \ [\alpha] \in F(G)(X)$$

$$= [g \circ (f \circ \alpha)] = g_*([f \circ \alpha]) = g_*(f_*([\alpha])) = (g_* \circ f_*)([\alpha]).$$

Thus $\forall \ [\alpha] \in F(G)(X)$, $(g \circ f)_* = g_* \circ f_*$.

Since $g \circ f \simeq I_G$, we have $(g \circ f)_* = I_{G*}$, by (i) $\Rightarrow$ $g_* \circ f_* = Id_{F(G)(X)}$, by (ii) and (iii) i.e. $g_* \circ f_* = Id$.

Again since $f \circ g \simeq I_H$, we have similarly

$$f_* \circ g_* = Id.$$

Since f_* is a homomorphism and $g_* \circ f_* = Id \ \Rightarrow \ f_*$ is a monomorphism. Again since f_* is a homomorphism and $g_* \circ f_* = Id \ \Rightarrow \ f_*$ is a epimorphism. Thus f_* is an isomorphism and g_* as its inverse.

Therefore the groups $F(G)(X)$ and $F(H)(X)$ are isomorphic.

Lemma 2.9. *Let G be a topological group and X, Y be two pointed topological spaces such that X and Y belong to the same homotopy type. Then the groups $F(G)(X)$ and $F(G)(Y)$ are isomorphic, where $F(G)$ is a contravariant functor from Htp to Grp given in Theorem 2.5.*

Proof. Let X, Y be two pointed topological spaces having the same homotopy type, then $\exists$ base point preserving continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq I_Y$ and $g \circ f \simeq I_X$, where $I_X : X \to X$ and $I_Y : Y \to Y$ are identity maps.

Define $f^* : F(G)(Y) \to F(G)(X)$ by

$$g^*([\alpha]) = [\alpha \circ f], \ \forall \ [\alpha] \in F(G)(Y).$$

Using Theorem 2.3 and Theorem 2.5, f^* is a homomorphism from $F(G)(Y)$ to $F(G)(X)$.

Then f^* satisfies the following properties:

(i) if $f \simeq g \ \Rightarrow \ f^* = g^*$;

(ii) $I_X : X \to X \ \Rightarrow \ I_X^* = Id_{F(G)(X)}$,

for $I_X^* : F(G)(X) \to F(G)(X)$ defined by

$$I_X^*([\alpha]) = [\alpha \circ I_X] = [\alpha], \ \forall \ [\alpha] \in F(G)(X)$$

i.e. $I_X^* = Id_{F(G)(X)}$

(iii) $(g \circ f)^* = f^* \circ g^*$,

for $(g \circ f)^* : F(G)(X) \to F(G)(X)$, defined by

$$(g \circ f)^*([\alpha]) = [\alpha \circ (g \circ f)], \ \forall \ [\alpha] \in F(G)(X) = [(\alpha \circ g) \circ f] =$$

$$= f^*([\alpha \circ g]) = f^*(g^*([\alpha])) = (f^* \circ g^*)([\alpha]).$$

Thus $\forall \ [\alpha] \in F(G)(X)$, $(g \circ f)^* = f^* \circ g^*$.

Since $g \circ f \simeq I_X$, we have $(g \circ f)^* = I_X^*$, by (i) $\Rightarrow$ $f^* \circ g^* = Id_{F(G)(X)}$, by (ii) and (iii) i.e. $f^* \circ g^* = Id$.

Again since $f \circ g \simeq I_Y$, we have similarly

$$g^* \circ f^* = Id.$$

Since f^* is a homomorphism and $g^* \circ f^* = Id \ \Rightarrow \ f^*$ is a monomorphism.

Again since f^* is a homomorphism and $f^* \circ g^* = Id \ \Rightarrow \ f^*$ is a epimorphism.

Therefore f^* is an isomorphism and g^* as its inverse.

Thus the groups $F(G)(X)$ and $F(G)(Y)$ are isomorphic.

Theorem 2.10. *For a given topological group G there always exists a contravariant functor $F(G) : Htp \to Grp$ such that $F(G)$ is homotopy type invariant.*

Proof. Using Lemma 2.9, it follows that $F(G)$ is a homotopy type invariant functor in th sense that if X and Y are the same homotopy type then the groups $F(G)(X)$ and $D(G)(Y)$ are isomorphic.

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