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KELVIN-HELMHOLTZ INSTABILITY OF RIVLIN-ERICKSEN VISCOELASTIC FLUID IN POROUS MEDIUM

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Abstract. Kelvin-Helmholtz instability of Rivlin-Ericksen elastico-viscous fluid in porous medium is considered. The case of two uniform streaming fluids separated by a horizontal boundary is considered. It is found that for the special case when perturbations in the direction of streaming are ignored, perturbation transverse to the direction of streaming are found to be unnafected by the presence of streaming. In every other direction, a minimum value of wave-number has been found and the system is unstable for all wave-numbers greater than this minimum wave number.

1. **Introduction**

When two superposed fluids flow one over the other with a relative horizontal velocity, the instability of the plane interface between the two fluids, when it occurs in this instance, is known as 'Kelvin-Helmholtz instability'. The instability of the plane interface separating two uniform superposed streaming fluids, under varying assumptions of hydrodynamics, has been discussed in the celebrated monograph by Chandrasekhar [1]. The experimental observation of the Kelvin-Helmholtz instability has been given by Francis [2]. The medium has been assumed to be non-porous.

With the growing importance of viscoelastic fluids in modern technology and industries and the investigations on such fluids are desirable. The Rivlin-Ericksen fluid is one such viscoelastic fluid. Many research workers have paid their attention towards the study of Rivlin-Ericksen fluid. Johri [3] has discussed the viscoelastic Rivlin-Ericksen incompressible fluid under time-dependent pressure gradient. Sisodia and Gupta [4] and Srivastava and Singh [5] have studied the unsteady flow of a dusty

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elastico-viscous Rivlin-Ericksen fluid through channel of different cross-sections in the present of the time dependent pressure gradient. Recently, Sharma and Kumar [6] have studied the thermal instability of a layer of Rivlin-Ericksen elastico-viscous fluid acted on by a uniform rotation and found that rotation has a stabilizing effect and introduces oscillatory modes in the system.

The flow through a porous medium has been of considerable interest in recent years particularly among geophysical fluid dynamicists. An example in the geophysical context is the recovery of crude oil from the pores of reservoir rocks. A great number of applications in geophysics may be found in a recent book by Phillips [7]. The gross effect when the fluid slowly percolates through the pores of the rock is given by Darcy's law. As a result, the usual viscous term in the equation of motion of Rivlin-Ericksen fluid is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu+\mu'\frac{\partial}{\partial t}\right)\overrightarrow{q}\right]$, where μ and μ' are the viscosity and viscoelasticity of the Rivlin-Ericksen fluid, k_1 is the medium permeability and $\overrightarrow{q}$ is the Darcian (filter) velocity of the fluid. Generally, it is accepted that comets consists of a dusty 'snowball' of a mixture of frozen gases which, in the process of their journey, changes from solid to gas and vice-versa. The physical properties of comets, meteorites and interplanetary dust strongly suggest the importance of porosity in astrophysical contex (McDonnel [8]). The instability of the plane interface between two uniform superposed and streaming fluids through porous medium has been investigated by Sharma and Spanos [9]. More recently, Sharma et al. [10] have studied the thermosolutal convection in Rivlin-Ericksen fluid in porous medium in the presence of uniform vertical magnetic field.

Keeping in mind the importance of non-Newtonian fluids in modern technology and industries and various applications mentioned above, Kelvin-Helmholtz instability of Rivlin-Ericksen viscoelastic fluid in porous medium has been considered in the present paper.

2. **Formulation of the problem and perturbation equations**

The initial stationary state, whose stability we wish to examine is that of an incompressible elastico-viscous Rivlin-Ericksen fluid in which there is a horizontal streaming in the x-direction with velocity $U(z)$ through a homogeneous, isotropic porous medium. The character of the equilibrium of this initial state is determined by supposing that the system is slightly disturbed and then following its further evolution.

Let $p, \rho, g, v, v', \overrightarrow{q}(U(z), 0, 0)$ denote, respectively, the pressure, density, acceleration due to gravity, kinematic viscosity, kinematic viscolasticity, and velocity of Rivlin-Ericksen viscoelastic fluid. This fluid layer is assumed to be flowing through an isotropic and homogeneous porous medium of porosity ε and medium permeability k_1 and interfacial tension effect is ignored. Then the equations of motion, continuity and incompressibility for the Rivlin-Ericksen elastico-viscous fluid through a porous medium are given by

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \overrightarrow{q}}{\partial t} + \frac{1}{\varepsilon}(\overrightarrow{q} \cdot \nabla)\overrightarrow{q}\right] = -\nabla p + \rho \overrightarrow{g} - \frac{\rho}{k_1}\left(v + v'\frac{\partial}{\partial t}\right)\overrightarrow{q}, \tag{1}$$

$$\nabla \cdot \overrightarrow{q} = 0, \tag{2}$$

$$\varepsilon\frac{\partial \rho}{\partial t} + (\overrightarrow{q} \cdot \nabla)\rho = 0. \tag{3}$$

Let $\delta p, \delta\rho$ and $\overrightarrow{u}(u, v, w)$ denote the perturbations in pressure p, density ρ and velocity $\overrightarrow{q}(U(z), 0, 0)$ respectively. Then, the linearized perturbation equations of fluid layer become

$$\frac{\rho}{\varepsilon}\left[\frac{\partial \overrightarrow{u}}{\partial t} + \frac{1}{\varepsilon}(\overrightarrow{q} \cdot \nabla)\overrightarrow{u} + \frac{1}{\varepsilon}(\overrightarrow{u} \cdot \nabla)\overrightarrow{q}\widehat{i}\right] = -\nabla\delta p + \overrightarrow{g}\delta p - \frac{\rho}{k_1}\left(v + v'\frac{\partial}{\partial t}\right)\overrightarrow{u}, \tag{4}$$

$$\nabla \cdot \overrightarrow{u} = 0, \tag{5}$$

$$\left[\varepsilon\frac{\partial}{\partial t} + (\overrightarrow{q} \cdot \nabla)\right]\delta p = -w\frac{d\rho}{dz}. \tag{6}$$

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is of the form

$$\exp[i(k_x x + k_y y + nt)], \tag{7}$$

where n is the growth rate, $k = (k_x^2 + k_y^2)^{1/2}$ is the resultant wave number and k_x, k_y are horizontal wave numbers.

Substituting for $\delta\rho$, Eq.(4) with the help of Eqs.(5),(6) and expression (7) yields

$$\left[\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right]\overrightarrow{u} + \frac{\rho}{\varepsilon^2}w(DU)\widehat{i} = -\nabla\delta p + i\overrightarrow{g}\frac{w(D\rho)}{\varepsilon n + k_x U}, \tag{8}$$

where $\widehat{i}$ is unit vector in the x-direction and $D = d/dz$.

Writing the three component equations of (8) and eliminating u, v and δp with the help of (5), we obtain

$$D\left[\left\{\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right\} Dw - \frac{ik_x\rho}{\varepsilon^2}(DU)w\right] -$$

$$-k^2\left[\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right] w = igk^2(D\rho)\frac{w}{\varepsilon n + k_x U}. \tag{9}$$

3. **Two uniform streaming fluids separated by a horizontal boundary**

Consider the case when two superposed streaming fluids of uniform densities ρ_1 and ρ_2, uniform viscosities μ_1 and μ_2 and uniform viscoelasticities μ_1' and μ_2' are separated by a horizontal boundary at $z = 0$. The subscript 1 and 2 distinguish the lower and the upper fluids respectively.

The density ρ_2 of the upper fluid is taken to be less than the density ρ_1 of the lower fluid so that, in the absence of streaming, the configuration is stable, and the porous medium throughout is assumed to be isotropic and homogeneous. Let the two fluids be streaming with constant velocities U_1 and U_2. Then in each of the two regions of constant ρ, μ, μ' and U, Eq.(9) reduces to

$$(D^2 - k^2)w = 0. \tag{10}$$

The boundary conditions to be satisfied here are:

(a) Since U is discontinuous at $z = 0$, the uniqueness of the normal displacement of any point on the interface, according to Eq.(8), implies that

$$\frac{w}{\varepsilon n + k_x U}, \tag{11}$$

must be continuous at an interface.

(b) Integrating Eq.(9) between $0 - \eta$ and $0 + \eta$ and passing to the limit $\eta = 0$, we obtain, in view of (11), the jump condition

$$\Delta_0\left[\left\{\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right\} Dw - \frac{ik_x\rho}{\varepsilon^2}(DU)w\right] = igk^2\Delta_0(\rho)\left(\frac{w}{\varepsilon n + k_x U}\right)_0 \tag{12}$$

(for $z = 0$) while the equation valid everywhere else ($z \neq 0$) is

$$D\left[\left\{\frac{i\rho}{\varepsilon^2}(\varepsilon n + k_x U) + \frac{\rho}{k_1}(v + inv')\right\} Dw - \frac{ik_x\rho}{\varepsilon^2}(DU)w\right] -$$

$$-k^2\left[\frac{i\rho}{\varepsilon^2}(\varepsilon n+k_xU)+\frac{\rho}{k_1}(v+inv')\right]w=igk^2(D\rho)\frac{w}{\varepsilon n+k_xU}. \tag{13}$$

Here $\Delta_0(f)=f(z_0+0)-f(z_0-0)$ is the jump which a quantity experiences at the interface $z=z_0$; and the subscript 0 distinguish the value a quantity, known to be continuous at an interface, takes at the interface $z=z_0$.

The general solution of Eq.(10) is a linear combination of the integrals e^{+kz} and e^{-kz}. Since $\dfrac{w}{\varepsilon n+k_xU}$ must be continuous on the surface $z=0$ and w cannot increase exponentially on either side of the interface, the solutions appropriate for two regions are

$$w_1=A(\varepsilon n+k_xU_1)e^{+kz},\quad (z<0) \tag{14}$$

$$w_2=A(\varepsilon n+k_xU_2)e^{-kz},\quad (z>0). \tag{15}$$

Applying the boundary condition (12) to the solutions (14)-(15), we obtain the dispersion relation

$$\begin{aligned}&\left[1+\frac{\varepsilon}{k_1}(\alpha_1v_1'+\alpha_2v_2')\right]n^2+\\&+\left[\frac{2k_x}{\varepsilon}(\alpha_1U_1+\alpha_2U_2)+\frac{k_x}{k_1}(\alpha_1v_1'U_1+\alpha_2v_2'U_2)-\frac{i\varepsilon}{k_1}(\alpha_1v_1+\alpha_2v_2)\right]n+\\&+\left[\frac{k^2}{\varepsilon^2}(\alpha_1U_1^2+\alpha_2U_2^2)-\frac{ik_x}{k_1}(\alpha_1v_1U_1+\alpha_2v_2U_2)-gk(\alpha_1-\alpha_2)\right]=0,\end{aligned} \tag{16}$$

where

$$\alpha_{1,2}=\frac{\rho_{1,2}}{\rho_1+\rho_2},\quad v_{1,2}=\frac{\mu_{1,2}}{\rho_{1,2}},\quad v_{1,2}'=\frac{\mu_{1,2}'}{\rho_{1,2}}.$$

$v_1\left(=\dfrac{\mu_1}{\rho_1}\right)$, $v_1'\left(=\dfrac{\mu_1'}{\rho_1}\right)$, $v_2\left(=\dfrac{\mu_2}{\rho_2}\right)$ and $v_2'\left(=\dfrac{\mu_2'}{\rho_2}\right)$ are the kinematic viscosities and kinematic viscoelasticities of the lower and upper fluids respectively.

Equation (16) yields

$$\begin{aligned}in=&-\left[+\frac{\varepsilon}{k_1}(\alpha_1v_1+\alpha_2v_2)+\frac{2ik_x}{\varepsilon}(\alpha_1U_1+\alpha_2U_2)+\frac{ik_x}{k_1}(\alpha_1v_1'U_1+\alpha_2v_2'U_2)\right]\pm\\&\pm\left\{\left[\frac{\varepsilon}{k_1}(\alpha_1v_1+\alpha_2v_2)\right]^2-\frac{4ik_x\alpha_1\alpha_2}{k_1}(v_1-v_2)(U_1-U_2)+\right.\\&+\frac{4k_x^2\alpha_1\alpha_2}{\varepsilon k_1}(v_2'U_1-v_1'U_2)(U_1-U_2)-\frac{2i\varepsilon k_x}{k_1^2}[(\alpha_1^2v_1v_1'U_1+\alpha_2^2v_2v_2'U_2)+\\&+\alpha_1\alpha_2(v_1v_2'U_1+v_1'v_2U_2)+\alpha_1\alpha_2(U_1-U_2)(v_1v_2'-v_1'v_2)]+\\&+\left[\frac{k_x}{k_1}(\alpha_1v_1'U_1+\alpha_2v_2'U_2)\right]^2+\frac{4\alpha_1\alpha_2k_x^2}{\varepsilon^2}(U_1-U_2)^2-\end{aligned}$$

$$-4gk(\alpha_1-\alpha_2)\left[1+\frac{\varepsilon}{k_1}(\alpha_1 v_1'+\alpha_2 v_2')\right]\Bigg\}^{\frac{1}{2}}. \tag{17}$$

Some cases of interest are now considered.

(a) When $k_x = 0$, equation (17) yields

$$in = -\frac{\varepsilon}{k_1}(\alpha_1 v_1+\alpha_2 v_2) \pm \left\{\left[\frac{\varepsilon}{k_1}(\alpha_1 v_1+\alpha_2 v_2)\right]^2 - \right.$$

$$\left. -4gk(\alpha_1-\alpha_2)\left[1+\frac{\varepsilon}{k_1}(\alpha_1 v_1'+\alpha_2 v_2')\right]\right\}^{\frac{1}{2}}. \tag{18}$$

Here we assume kinematic viscosities v_1, v_2 and kinematic viscoelasticities v_1', v_2' of the two fluids to be equal i.e., $v_1 = v_2 = v$, $v_1' = v_2' = v'$. However, any of the essential features of the problem are not obscured by this simplifying assumption. Eq.(18), then, becomes

$$in = -\frac{\varepsilon v}{k_1} \pm \left[\left(\frac{\varepsilon v}{k_1}\right)^2 + 4gk(\alpha_2-\alpha_1)\left\{1+\frac{\varepsilon v'}{k_1}\right\}\right]^{\frac{1}{2}}. \tag{19}$$

(i) **Unstable case**

For the potentially unstable configuration ($\rho_2 > \rho_1$), it is evident from Eq.(19) that one of the values of in is positive which means that the perturbations grow with time and so the system is unstable.

(ii) **Stable case**

For the potentially stable configuration ($\rho_2 < \rho_1$), Eq.(19) yields that both the values of in are either real, negative or complex conjugates with negative real parts implying stability of the system.

It is interesting to note from above that for the special case when perturbations in the direction of streaming are ignored ($k_x = 0$), the system is unstable for potentially unstable configuration and the system is stable for potentially stable configuration and not depending upon kinematic viscoelasticity, medium porosity and medium permeability. This is in contrast to the case of Walters' viscoelastic fluid B', where the system can be stable or unstable depending upon kinematic viscoelaticity, medium porosity and medium permeability (Sharma et al. [11]).

It is also clear from Eq.(18), that for the special case when perturbations in the direction of streaming are ignored ($k_x = 0$), the perturbation transverse to the direction of streaming ($k_y \neq 0$) are unaffected by the presence of streaming.

(b) In every other direction, instability occurs when

$$\frac{\alpha_1 \alpha_2 k_x^2}{\varepsilon^2}(U_1 - U_2)^2 > gk(\alpha_1 - \alpha_2). \tag{20}$$

The kinematic viscosities v_1 and v_2 and the kinematic viscoelasticies v_1' and v_2' of two fluids here are assumed to be equal (let $v_1 = v_2 = v$, $v_1' = v_2' = v'$), but this simplifying assumption does not obscure any of the essential features of the problem.

Thus for a given difference in velocity $(U_1 - U_2)$ and for a given direction of the wave-vector $\overrightarrow{k}$, instability occurs for all wave numbers.

$$k > \left[\frac{g\varepsilon^2(\alpha_1 - \alpha_2)}{\alpha_1 \alpha_2 (U_1 - U_2)^2 \cos^2\theta}\right], \tag{21}$$

where θ is the angle between the direction of $\overrightarrow{k}(k_x, k_y, 0)$ and $\overrightarrow{U}(U, 0, 0)$, i.e. $k_x = k\cos\theta$. Hence, for a given velocity differences $(U_1 - U_2)$, instability occurs for the least wave number when $\overrightarrow{k}$ is in the direction of $\overrightarrow{U}$ and this minimum wave number; k_{min}, is given by

$$k_{min} = \left[\frac{g\varepsilon^2(\alpha_1 - \alpha_2)}{\alpha_1 \alpha_2 (U_1 - U_2)^2}\right]. \tag{22}$$

For $k > k_{min}$, the system is unstable.

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