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ON A CERTAIN FAMILIES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. We introduce the subclass $T_j(n,m,\lambda,\alpha)$ of analytic functions with negative coefficients defined by Salagean operators D^n and D^{n+m}. In this paper we give some properties of functions in the class $T_j(n,m,\lambda,\alpha)$ and obtain numerous sharp results including (for example) coefficient estimates, distortion theorems, closure theorems and modified Hadamard products of several functions belonging to the class $T_j(n,m,\lambda,\alpha)$. We also obtain radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class $T_j(n,m,\lambda,\alpha)$ and consider integral operators associated with functions belonging to the class $T_j(n,m,\lambda,\alpha)$.

1. Introduction

Let $A(j)$ denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in N = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. For a function $f(z)$ in $A(j)$, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z) \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in N). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [5]. With the help of the differential operator D^n, we say that a function $f(z)$ belonging to $A(j)$ is in

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the class $S_j(n,m,\lambda,\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{(1-\lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1-\lambda)D^n f(z) + \lambda D^{n+m} f(z)}\right\} > \alpha \quad (n,m \in N_0 = N \cup \{0\}) \tag{1.5}$$

for some α $(0 \le \alpha < 1)$ and λ $(0 \le \lambda \le 1)$, and for all $z \in U$. The operator D^{n+m} was studied by Sekine [7] and Aouf and Salagean [2].

Let $T(j)$ denote the subclass of $A(j)$ consisting of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \ge 0;\ j \in N). \tag{1.6}$$

Further, we define the class $T_j(n,m,\lambda,\alpha)$ by

$$T_j(n,m,\lambda,\alpha) = S_j(n,m,\lambda,\alpha) \cap T(j). \tag{1.7}$$

We note that by specializing the parameters j,n,m,λ and α, we obtain the following subclasses studied by various authors:

(i) $T_j(n,1,\lambda,\alpha) = P(j,\lambda,\alpha,n)$, $T_j(n,m,0,\alpha) = P(j,\alpha,n)$ and $T_j(n,1,1,\alpha) = P(j,\alpha,n+1)$ (Aouf and Srivastava [3]);

(ii) $T_j(0,1,\lambda,\alpha) = P(j,\lambda,\alpha)$ (Altintas [1]);

(iii) $T_j(0,0,0,\alpha) = T_\alpha(j)$ and $T_j(0,1,1,\alpha) = T_j(1,0,1,\alpha) = C_\alpha(j)$ (Chatterjea [4] and Srivastava et al. [9]);

(v) $T_j(n,m,1,\alpha) = T_j(n,m,\alpha)$, where $T_j(n,m,\alpha)$ represents the class of functions $f(z) \in T(j)$ satisfying the condition

$$\operatorname{Re}\left\{\frac{z(D^{n+m} f(z))'}{D^{n+m} f(z)}\right\} > \alpha \quad (n,m \in N_0;\ 0 \le \alpha < 1;\ z \in U); \tag{1.8}$$

(iv) $T_1(0,0,0,\alpha) = T^*(\alpha)$ and $T_1(0,1,1,\alpha) = T_1(1,0,1,\alpha) = C(\alpha)$ (Silverman [8]).

2. Coefficient estimates and other properties of the class $T_j(n,m,\lambda,\alpha)$

Theorem 1. *Let the function $f(z)$ be defined by (1.6). Then $f(z) \in T_j(n,m,\lambda,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]a_k \le 1-\alpha. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\left|\frac{(1-\lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1-\lambda)D^n f(z) + \lambda D^{n+m} f(z)} - 1\right| \leq$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n(k-1)[1+(k^m-1)\lambda]a_k|z|^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k|z|^{k-1}} \leq$$

$$\leq \frac{\sum_{k=j+1}^{\infty} k^n(k-1)[1+(k^m-1)\lambda]a_k}{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k} \leq 1-\alpha.$$

This show that the values of the function

$$\Phi(z) = \frac{(1-\lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1-\lambda)D^n f(z) + \lambda D^{n+m} f(z)} \tag{2.2}$$

lie in a circle which is centered at $w = 1$ and whose radius is $1-\alpha$. Hence $f(z)$ satisfies the condition (1.5).

Conversely, assume that the function $f(z)$ is in the class $T_j(n, m, \lambda, \alpha)$. Then we have

$$\mathrm{Re}\left\{\frac{(1-\lambda)z(D^n f(z))' + \lambda z(D^{n+m} f(z))'}{(1-\lambda)D^n f(z) + \lambda D^{n+m} f(z)}\right\} =$$

$$= \mathrm{Re}\left\{\frac{1-\sum_{k=j+1}^{\infty} k^{n+1}[1+(k^m-1)\lambda]a_k z^{k-1}}{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k z^{k-1}}\right\} > \alpha, \tag{2.3}$$

for some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda \leq 1)$, $n, m \in N_0$ and for all $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we can see that

$$1-\sum_{k=j+1}^{\infty} k^{n+1}[1+(k^m-1)\lambda]a_k \geq \alpha\left\{1-\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k\right\}. \tag{2.4}$$

Thus we have the inequality (2.1).

Finally, the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]}z^k \quad (k \geq j+1;\ j \in N) \tag{2.5}$$

is an extremal function for the assertion of Theorem 1.

Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then*

$$a_k \leq \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \quad (k \geq j+1). \tag{2.6}$$

The equality in (2.6) is attained for the function $f(z)$ given by (2.5).

Theorem 2. *Let $0 \leq \alpha_1 \leq \alpha_2 < 1,\ 0 \leq \lambda \leq 1,\ j \in N$ and $n,m \in N_0$. Then*

$$T_j(n,m,\lambda,\alpha_1) \supseteq T_j(n,m,\lambda,\alpha_2). \tag{2.7}$$

Proof. Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha_2)$ and let $\alpha_1 = \alpha_2 - \delta$. Then, by Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]a_k \leq 1-\alpha_2 \tag{2.8}$$

and

$$\sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k \leq \frac{1-\alpha_2}{j+1-\alpha_2} < 1. \tag{2.9}$$

Consequently,

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha_1)[1+(k^m-1)\lambda]a_k = \sum_{k=j+1}^{\infty} k^n(k-\alpha_2)[1+(k^m-1)\lambda]a_k +$$

$$+\delta \sum_{k=j+1}^{\infty} k^n[1+(k^m-1)\lambda]a_k \leq 1-\alpha_1. \tag{2.10}$$

This completes the proof of Theorem 2 with the aid of Theorem 1.

Theorem 3. *Let $0 \leq \alpha < 1,\ 0 \leq \lambda_1 \leq \lambda_2 \leq 1,\ j \in N$ and $n,m \in N_0$. Then*

$$T_j(n,m,\lambda_1,\alpha) \supseteq T_j(n,m,\lambda_2,\alpha). \tag{2.11}$$

Proof. It follows from Theorem 1 that

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda_1]a_k \leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda_2]a_k \leq 1-\alpha$$

for $f(z) \in T_j(n,m,\lambda_2,\alpha)$.

Theorem 4. For $0 \leq \alpha < 1,\ 0 \leq \lambda \leq 1,\ j \in N$ and $n,m \in N_0$,

$$T_j(n+1,m,\lambda,\alpha) \subseteq T_j(n,m,\lambda,\alpha). \tag{2.12}$$

The proof of Theorem 4 follows also from Theorem 1.

3. Growth and distortion theorems

Theorem 5. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then for $|z| = r < 1$,*

$$|D^i f(z)| \geq r - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m - 1]\lambda]} r^{j+1} \tag{3.1}$$

and

$$|D^i f(z)| \leq r + \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m - 1]\lambda]} r^{j+1} \tag{3.2}$$

for $z \in U$ and $0 \leq i \leq n$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(j+1)^{n}(j+1-\alpha)[1+[(j+1)^m - 1]\lambda]} z^{j+1} \quad (z = \pm r). \tag{3.3}$$

Proof. Note that $f(z) \in T_j(n, m, \lambda, \alpha)$ if and only if

$$D^i f(z) \in T_j(n-i, m, \lambda, \alpha)$$

and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \tag{3.4}$$

By Theorem 1, we know that

$$(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m - 1]\lambda] \sum_{k=j+1}^{\infty} k^i a_k \leq$$

$$\leq \sum_{k=j+1}^{\infty} k^n (k-\alpha)[1+(k^m-1)\lambda] a_k \leq 1-\alpha, \tag{3.5}$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m - 1]\lambda]}. \tag{3.6}$$

The assertions (3.1) and (3.2) of Theorem 5 would now follow readily from (3.4) and (3.6).

Finally, we note that the equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1}. \tag{3.7}$$

This completes the proof of Theorem 5.

Corollary 2. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then, for $|z| = r < 1$,*

$$|f(z)| \geq r - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1} \tag{3.8}$$

and

$$|f(z)| \leq r + \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^{j+1}. \tag{3.9}$$

The equalities in (3.8) and (3.9) are attained for the function $f(z)$ given by (3.3).

Proof. Taking $i = 0$ in Theorem 5, we immediately obtain (3.8) and (3.9).

Corollary 3. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then for $|z| = r < 1$,*

$$|f'(z)| \geq \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^j \tag{3.10}$$

and

$$|f'(z)| \leq 1 + \frac{1-\alpha}{(j+1)^{n-i}(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} r^j \quad (z \in U). \tag{3.11}$$

The equalities in (3.10) and (3.11) are attained for the function $f(z)$ given by (3.3).

Proof. Setting $i = 1$ in Theorem 5, and making use of the definition (1.3), we arrive at Corollary 3.

4. **Convex linear combinations**

In this section, we shall prove that the class $T_j(n,m,\lambda,\alpha)$ is closed under convex linear combinations.

Theorem 6. *$T_j(n,m,\lambda,\alpha)$ is a convex set.*

Proof. Let the functions

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0;\ v = 1,2) \tag{4.1}$$

be in the class $T_j(n, m, \lambda, \alpha)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \quad (0 \le \mu \le 1) \tag{4.2}$$

is also in the class $T_j(n, m, \lambda, \alpha)$. Since, for $0 \le \mu \le 1$,

$$h(z) = z - \sum_{k=j+1}^{\infty} [\mu a_{k,1} + (1-\mu) a_{k,2}] z^k, \tag{4.3}$$

with the aid of Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n (k-\alpha)[1 + (k^m - 1)\lambda][\mu a_{k,1} + (1-\mu) a_{k,2}] \le 1 - \alpha, \tag{4.4}$$

which implies that $f(z) \in T_j(n, m, \lambda, \alpha)$. Hence $T_j(n, m, \lambda, \alpha)$ is a convex set.

Theorem 7. *Let*

$$f_j(z) = z \tag{4.5}$$

and

$$f_k(z) = z - \frac{1-\alpha}{k^n (k-\alpha)[1 + (k^m - 1)\lambda]} z^k \quad (k \ge j+1;\ n, m \in N_0) \tag{4.6}$$

for $0 \le \alpha < 1$ and $0 \le \lambda \le 1$. Then $f(z)$ is in the class $T_j(n, m, \lambda, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z), \tag{4.7}$$

where

$$\mu_k \ge 0 \ (k \ge j) \quad \text{and} \quad \sum_{k=j}^{\infty} \mu_k = 1. \tag{4.8}$$

Proof. Assume that

$$f(z) = \sum_{k=j}^{\infty} \mu_k f_k(z) =$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{k^n (k-\alpha)[1 + (k^m - 1)\lambda]} \mu_k z^k. \tag{4.9}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \frac{k^n (k-\alpha)[1 + (k^m - 1)\lambda]}{1-\alpha} \cdot \frac{1-\alpha}{k^n (k-\alpha)[1 + (k^m - 1)\lambda]} \mu_k =$$

$$= \sum_{k=j+1}^{\infty} \mu_k = 1 - \mu_j \le 1. \tag{4.10}$$

So, by Theorem 1, $f(z) \in T_j(n, m, \lambda, \alpha)$.

Conversely, assume that the function $f(z)$ defined by (1.6) belongs to the class $T_j(n, m, \lambda, \alpha)$. Then

$$a_k \leq \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \quad (k \geq j+1;\ n, m \in N_0). \tag{4.11}$$

Setting

$$\mu_k = \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_k \quad (k \geq j+1;\ n, m \in N_0) \tag{4.12}$$

and

$$\mu_j = 1 - \sum_{k=j+1}^{\infty} \mu_k, \tag{4.13}$$

we can see that $f(z)$ can be expressed in the form (4.7). This completes the proof of Theorem 7.

5. Radii of close-to-convexity, starlikeness, and convexity

Theorem 8. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n, m, \lambda, \alpha)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$, where*

$$r_1 = r_1(n, m, \lambda, \alpha, \rho) = \inf_k \left[\frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.1}$$

The result is sharp, the extremal function $f(z)$ begin given by (2.5).

Proof. We must show that

$$|f'(z) - 1| \leq 1 - \rho \text{ for } |z| < r_1(n, m, \lambda, \alpha, \rho),$$

where $r_1(n, m, \lambda, \alpha, \rho)$ is given by (5.1). Indeed we find from the definition (1.6) that

$$|f'(z) - 1| \leq \sum_{k=j+1}^{\infty} k a_k |z|^{k-1}.$$

Thus

$$|f'(z) - 1| \leq 1 - \rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \leq 1. \tag{5.2}$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.3}$$

Theorem 8 follows easily from (5.3).

Theorem 9. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(n,m,\lambda,\alpha,\rho) = \inf_f \left[\frac{(1-\rho)k^n(k-\alpha)[1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.4}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho \text{ for } |z| < r_2(n,m,\lambda,\alpha,\rho),$$

where $r_2(n,m,\lambda,\alpha,\rho)$ is given by (5.4). Indeed we find, again from the definition (1.6), that

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{k=j+1}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}a_k|z|^{k-1}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq 1-\rho$$

if

$$\sum_{k=j+1}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k|z|^{k-1} \leq 1. \tag{5.5}$$

But, by Theorem 1, (5.5) will be if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1} \leq \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)k^n(k-\alpha)[1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{5.6}$$

Theorem 9 follows easily from (5.6).

Corollary 4. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$. Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = r_3(n,m,\lambda,\alpha,\rho) = \inf_k \left[\frac{(1-\rho)k^{n-1}(k-\alpha)[1+(k^m-1)\lambda]}{(k-\rho)(1-\alpha)} \right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{5.7}$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

6. Modified Hadamard products

Let the functions $f_v(z)$ $(v = 1,2)$ be defined by (4.1). The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1 * f_2(z) = z - \sum_{k=j+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{6.1}$$

Theorem 10. *Let each of the functions $f_v(z)$ $(v = 1,2)$ defined by (4.1) be in the class $T_j(n,m,\lambda,\alpha)$. Then*

$$f_1 * f_2(z) \in T_j(n,m,\lambda,\beta(j,n,m,\lambda,\alpha)),$$

where

$$\beta(j,n,m,\lambda,\alpha) = 1 - \frac{j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+\lambda[(j+1)^m-1]]-(1-\alpha)^2}. \tag{6.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [6], we need to find the largest $\beta = \beta(j,n,m,\lambda,\alpha)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\beta)[1+(k^m-1)\lambda]}{1-\beta} a_{k,1} a_{k,2} \le 1. \tag{6.3}$$

Since

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,1} \le 1 \tag{6.4}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_{k,2} \le 1, \tag{6.5}$$

by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \le 1. \tag{6.6}$$

Thus it is sufficient to show that

$$\frac{k^n(k-\beta)[1+(k^m-1)\lambda]}{1-\beta}a_{k,1}a_{k,2} \leq \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\sqrt{a_{k,1}a_{k,2}}\ (k \geq j+1), \tag{6.7}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq j+1). \tag{6.8}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \quad (k \geq j+1). \tag{6.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n(k-\alpha)[1+(k^m-1)\lambda]} \leq \frac{(k-\alpha)(1-\beta)}{(k-\beta)(1-\alpha)} \quad (k \geq j+1), \tag{6.10}$$

or, equivalently, that

$$\beta \leq 1 - \frac{(k-1)(1-\alpha)^2}{k^n(k-\alpha)^2[1+(k^m-1)\lambda]-(1-\alpha)^2} \quad (k \geq j+1). \tag{6.11}$$

Since

$$A(k) = 1 - \frac{(k-1)(1-\alpha)^2}{k^n(k-\alpha)^2[1+(k^m-1)\lambda]-(1-\alpha)^2} \tag{6.12}$$

is an increasing function of k $(k \geq j+1)$, letting $k = j+1$ in (6.12) we obtain

$$\beta \leq A(j+1) = \frac{j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+[(j+1)^m-1]\lambda]-(1-\alpha)^2}, \tag{6.13}$$

which proves the main assertion of Theorem 10.

Finally, by taking the functions

$$f_v(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]}z^{j+1} \quad (v=1,2), \tag{6.14}$$

we can see that the result is sharp,

Theorem 11. *Let* $f_1(z) \in T_j(n,m,\lambda,\alpha)$ *and* $f_2(z) \in T_j(n,m,\lambda,\gamma)$. *then*

$$f_1 * f_2(z) \in T_j(n,m,\lambda,\xi(j,n,m,\lambda,\alpha,\gamma)),$$

where

$$\xi(j,n,m,\lambda,\alpha,\gamma) = \tag{6.15}$$

$$= 1 - \frac{j(1-\alpha)(1-\gamma)}{(j+1)^n(j+1-\alpha)(j+1-\gamma)[1+[(j+1)^m-1]\lambda]-(1-\alpha)(1-\gamma)}.$$

The result is best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1} \tag{6.16}$$

and

$$f_2(z) = z - \frac{1-\gamma}{(j+1)^n(j+1-\gamma)[1+[(j+1)^m-1]\lambda]} z^{j+1}. \tag{6.17}$$

Proof. Proceeding as in the proof of Theorem 10, we get

$$\xi \leq 1 - \frac{(k-1)(1-\alpha)(1-\gamma)}{k^n(k-\alpha)(k-\gamma)[1+(k^m-1)\lambda]-(1-\alpha)(1-\gamma)} \quad (k \geq j+1). \tag{6.18}$$

Since the right hand side of (6.18) is an increasing function of k, setting $k = j+1$ in (6.18) we obtain (6.15). This completes the proof of Theorem 11.

Corollary 5. *Let the functions $f_v(z)$ defined by*

$$f_v(z) = z - \sum_{k=j+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0,\ v = 1,2,3) \tag{6.19}$$

be in the class $T_j(n,m,\lambda,\alpha)$. Then

$$f_1 * f_2 * f_3(z) \in T_j(n,m,\lambda,\delta(j,n,m,\lambda,\alpha)),$$

where

$$\delta(j,n,m,\lambda,\alpha) = 1 - \frac{j(1-\alpha)^3}{(j+1)^{2n}(j+1-\alpha)^3[1+[(j+1)^m-1]\lambda]^2-(1-\alpha)^3}. \tag{6.20}$$

The result is best possible for the functions

$$f_v(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1+[(j+1)^m-1]\lambda]} z^{j+1} \quad (v = 1,2,3). \tag{6.21}$$

Proof. From Theorem 10, we have

$$f_1 * f_2(z) \in T_j(n,m,\lambda,\beta(j,n,m,\lambda,\alpha)),$$

where β is given by (6.2). Now, using Theorem 11, we get

$$f_1 * f_2 * f_3(z) \in T_j(n,m,\lambda,\delta(j,n,m,\lambda,\alpha)),$$

where

$$\delta(j,n,m,\lambda,\alpha) =$$
$$= 1 - \frac{j(1-\alpha)(1-\beta)}{(j+1)^n(j+1-\alpha)(j+1-\beta)[1+[(j+1)^m-1]\lambda]-(1-\alpha)(1-\beta)} =$$

$$= 1 - \frac{j(1-\alpha)^3}{(j+1)^{2n}(j+1-\alpha)^3[1+[(j+1)^m-1]\lambda]-(1-\alpha)^3}.$$

This completes the proof of Corollary 5.

Theorem 12. *Let the functions $f_v(z)$ $(v=1,2)$ defined by (4.1) be in the class $T_j(n,m,\lambda,\alpha)$, then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k \tag{6.22}$$

belongs to the class $T_j(n,m,\lambda,\eta(j,n,m,\lambda,\alpha))$, where

$$\eta(j,n,m,\lambda,\alpha) = 1 - \frac{2j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+[(j+1)^m-1]\lambda]-2(1-\alpha)^2}. \tag{6.23}$$

The result is sharp for the functions $f_v(z)$ $(v=1,2)$ defined by (6.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=j+1}^{\infty}\left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 a_{k,1}^2 \leq \tag{6.24}$$

$$\leq \left[\sum_{k=j+1}^{\infty}\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}a_{k,1}\right]^2 \leq 1$$

and

$$\sum_{k=j+1}^{\infty}\left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 a_{k,2}^2 \leq \tag{6.25}$$

$$\leq \left[\sum_{k=j+1}^{\infty}\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}a_{k,2}\right]^2 \leq 1.$$

It follows from (6.24) and (6.25) that

$$\sum_{k=j+1}^{\infty}\frac{1}{2}\left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \tag{6.26}$$

Therefore, we need to find the largest $\eta = \eta(j,n,m,\lambda,\alpha)$ such that

$$\frac{k^n(k-\eta)[1+(k^m-1)\lambda]}{1-\eta} \leq \frac{1}{2}\left[\frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha}\right]^2 \quad (k \geq j+1), \tag{6.27}$$

that is,

$$\eta \leq 1 - \frac{2(k-1)(1-\alpha)^2}{(k-\alpha)^2 k^n[1+(k^m-1)\lambda]-2(1-\alpha)^2} \quad (k \geq j+1). \tag{6.28}$$

Since

$$B(k) = 1 - \frac{2(k-1)(1-\alpha)^2}{k^n(k-\alpha)^2[1+(k^m-1)\lambda]-2(1-\alpha)^2}. \tag{6.29}$$

is an increasing function of k $(k \geq j+1)$, we readily have

$$\eta \leq B(j+1) = 1 - \frac{2j(1-\alpha)^2}{(j+1)^n(j+1-\alpha)^2[1+[(j+1)^m-1]\lambda]-2(1-\alpha)^2}, \quad (6.30)$$

and Theorem 12 follows at once.

7. **A family of integral operators**

Theorem 13. *Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\lambda,\alpha)$, and let c be a real number such that $c > -1$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (c > -1) \quad (7.1)$$

also belongs to the class $T_j(n,m,\lambda,\alpha)$.

Proof. From the representation (7.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k z^k,$$

where

$$b_k = \left(\frac{c+1}{c+k}\right)a_k.$$

Therefore, we have

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]b_k = \sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]\left(\frac{c+1}{c+k}\right)a_k \leq$$

$$\leq \sum_{k=j+1}^{\infty} k^n(k-\alpha)[1+(k^m-1)\lambda]a_k \leq 1-\alpha,$$

since $f(z) \in T_j(n,m,\lambda,\alpha)$. Hence, by Theorem 1, $F(z) \in T_j(n,m,\lambda,\alpha)$.

Theorem 14. *Let the function*

$$F(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0,\ j \in N)$$

be in the class $T_j(n,m,\lambda,\alpha)$, and let c be a real number such that $c > -1$. Then the function $f(z)$ given by (7.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \left[\frac{(k-\alpha)k^{n-1}[1+(k^m-1)\lambda](c+1)}{(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}} \quad (k \geq j+1). \quad (7.2)$$

The result is sharp.

Proof. From (7.1), we have

$$f(z) = \frac{z^{1-c}(z^c F(z))'}{c+1} = z - \sum_{k=j+1}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k.$$

In order to obtain the required result, it suffices to show that

$$|f'(z) - 1| < 1 \text{ whenever } |z| < R^*,$$

where R^* is given by (7.2). Now

$$|f'(z) - 1| \le \sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=j+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \tag{7.3}$$

But Theorem 1 confirms that

$$\sum_{k=j+1}^{\infty} \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha} a_k \le 1. \tag{7.4}$$

Hence (7.3) will be satisfied if

$$\frac{k(c+k)}{c+1}|z|^{k-1} < \frac{k^n(k-\alpha)[1+(k^m-1)\lambda]}{1-\alpha},$$

that is, if

$$|z| < \left[\frac{(k-\alpha)k^{n-1}[1+(k^m-1)\lambda](c+1)}{(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}} \quad (k \ge j+1). \tag{7.5}$$

Therefore, the function $f(z)$ given by (7.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(1-\alpha)(c+k)}{k^n(k-\alpha)[1+(k^m-1)\lambda](c+1)} z^k \quad (k \ge j+1). \tag{7.6}$$

References

[1] O. Altantas, *On s subclass of certain starlike functions with negative coefficients*, Math. Japon. 36(1991), 489-495.

[2] M.K. Aouf and G.S. Salagean, *Prestarlike functions with negative coefficients*, Rev. Roum. Math. Pures Appl. (to appear).

[3] M.K. Aouf and H.M. Srivastava, *Some families of starlike functions with negative coefficients*, J. Math. Anal. Appl. 203(1996), 762-790.

[4] S.K. Chatterjea, *On starlike functions*, J. Pure Math. 1(1981), 23-26.

[5] G.S. Salagean, *Subclasses of univalent function*, Complex Analysis - Fifth Romanian - Finnish Seminar Bucharest 1981, Proceedings, part 1, lect. Notes in Math. 1013, Springer Verlag 1983, 362-372.

[6] A. Schild and H. Silverman, *Convolutions of univalent function with negative coefficients*, Ann. Univ. Mariae Curie-Sklodowska, Sect.A, 29(1975), 99-107.
[7] T. Sekine, *Generalization of certain subclasses of analytic function*, Internat. J. Math. Sci. 10(1987), no.4, 725-732.
[8] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51(1975), 109-116.
[9] H.M. Srivastava, S. Owa and S.K. Chatterjea, *A note on certain classes of starlike functions*, Rend. Sem. Mat. Univ. Padova 77(1987), 115-124.

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