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# A NOTE ON STANDARD TOPOLOGICAL CONTEXTS WITH PSEUDOMETRIC

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**Abstract**. Standard topological contexts are a valuable tool in representing several classes of ordered algebraic structures. While investigating Contextual Topology, pseudometric contexts were introduced as a tool in approximating objects by their attributes. Here we describe the interaction between these two classes, i.e., pseudometric contexts and standard topological contexts, pointing out whether the Hartung duality extends in the case of metric lattices or not. Moreover, the meaning of being a contraction or being continuous in the case of multivalued pseudometric morphisms is investigated.

## 1. Introduction

Formal Concept Analysis was introduced first in an attempt of restructuring lattice theory (see [Wi82]). Since then, Formal Concept Analysis developed continuously to a theory of interpreting data by revealing the fundamental patterns of it. These patterns are then synthesized in a structure called **concept lattice**. Ten years later, standard topological contexts were introduced as a valuable tool in representing 0–1 lattices via Formal Concept Analysis ([Ha92]). This representation could be also considered as the first step in investigating the links between Topology and Formal Concept Analysis.

In [Sa00a] pseudometric and metric formal contexts were introduced as a generalization of the well known concepts of a metric on a set. By this generalization, the notion of metric extends on a formal context by the mathematization of a well known fact: Formal contexts are representing data sets. Usually, a data set is a record of several measurements or informations about a set of objects and a set of attributes of interest. These attributes are specific for the topic in study but some of these are more characteristic than others. (Pseudo)metric contexts, and uniform contexts as well, captures at best this phenomenon, i.e., that an attribute is more or less characteristic for an object as another one. For more informations, see for example [Sa00b].

This paper describes the links between standard topological contexts and pseudo metric contexts, investigating whether the well known duality between standard topological contexts and 0–1 lattices remains valid if we consider 0–1 pseudometric lattices. Moreover, we shall describe how some properties of 0–1 pseudometric lattices like being a contraction or being continuous reflects in the category of standard topological contexts with pseudometric.

## 2. **Basic Definitions and Results**

We briefly sketch the duality between bounded lattices and standard topological contexts developed in [Ha92] and [Ha93]. We recall some definitions and basic facts, for other definitions and results we refer to [GW96].

By $(X, \tau)$ we denote a topological space, where $X$ is the underlying set and $\mathcal{T}$ is the family of all closed sets of that space. We start with a triple $\mathbb{K}^{\mathcal{T}} := ((G, \rho), (M, \sigma), I)$ consisting of two topological spaces $(G, \rho), (M, \sigma)$ and a binary relation $I \subseteq G \times M$. For $A \subseteq G$ and $B \subseteq M$, we define two derivations by

$$A' := \{m \in M |\ gIm \text{ for all } g \in A\} \ and$$

$$B' := \{g \in G |\ gIm \text{ for all } m \in B\}.$$

These form a Galois-connection which gives rise to a complete lattice

$$\underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}}) := \{(A, B) |\ A \subseteq G, B \subseteq M, A' = B, B' = A\}$$

which is known as the **concept lattice** of the **context** $\mathbb{K}^{\mathcal{T}}$. The elements of $\underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}})$ are called (formal) **concepts**. If $(A, B)$ is a concept of $\mathbb{K}^{\mathcal{T}}$, the sets $A$ and $B$ are called the **extent** and the **intent** of the concept $(A, B)$. For two concepts, the relation subconcept–superconcept is given by

$$(A, B) \leq (C, D) \Leftrightarrow A \subseteq B (\Leftrightarrow B \supseteq B).$$

A **closed concept** is a concept consisting in each component of a closed set with respect to the corresponding topology. The set of all closed concepts is denoted by

$$\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) := \{(A, B) \in \underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}}) |\ A \in \rho \text{ and } B \in \sigma\}.$$

The triple $\mathbb{K}^{\mathcal{T}} := ((G, \rho), (M, \sigma), I)$ is called a **topological context** if the following two conditions are satisfied:

(i) $A \in \rho \Rightarrow A'' \in \rho; B \in \sigma \Rightarrow B'' \in \sigma$.

(ii) $\mathcal{S}_\rho := \{A \subseteq G |\ (A, A') \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})\}$ is a subbasis of $\rho$ and $\mathcal{S}_\sigma := \{B \subseteq M |\ (B, B') \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})\}$ is a subbasis of $\sigma$.

Under these assumptions, $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with the induced order is a 0–1 lattice in which infima and suprema can be described as follows

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, (B_1 \cup B_2)'');$$

$$(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2)'', B_1 \cap B_2).$$

For each $g \in G$, the concept $\gamma g := (g'', g')$ is called the **object concept** of $G$ and for each $m \in M$, the concept $\mu m := (m', m'')$ is called the **attribute concept** of $m$. We call a context **clarified** if $g, h \in G$ with $g' = h'$ implies $g = h$ and $m, n \in M$ with $m' = n'$ implies $m = n$. A clarified context is called **reduced** if each object concept is completely join-irreducible and each attribute concept is completely meet-irreducible. For each context $\mathbb{K} := (G, M, I)$, every $g \in G$ and $m \in M$, we define:

$$g \swarrow m \Leftrightarrow g \not{I} m \text{ and } (g' \subset h' \Rightarrow m \in h');$$

$$g \nearrow m \Leftrightarrow g \not{I} m \text{ and } (m' \subset n' \Rightarrow g \in n');$$

$$g \swarrow\!\!\!\nearrow m \Leftrightarrow g \swarrow m \text{ and } g \nearrow m.$$

We call two contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ **isomorphic** if there are bijective maps $\alpha : G_1 \to G_2$ and $\beta : M_1 \to M_2$ such that for all $g \in G_1$ and $m \in M_1$, the following condition is fulfilled:

$$g I_1 m \Leftrightarrow \alpha(g) I_2 \beta(m).$$

For each $H \subseteq G$ and $N \subseteq M$, the context $(H, N, I \cap (H \times N))$ is called a **subcontext** of $\mathbb{K}$. This subcontext is **compatible** if $(A, B) \in \underline{\mathcal{B}}(\mathbb{K})$ implies $(A \cap H, B \cap N) \in \underline{\mathcal{B}}(H, N, I \cap (H \times N))$.

**Proposition 2.1.** *A subcontext* $(H, N, I \cap (H \times N))$ *of* $\mathbb{K}$ *is compatible if and only if*

$$\Pi_{H,N} : \underline{\mathcal{B}}(\mathbb{K}) \to \underline{\mathcal{B}}(H, N, I \cap (H \times N)) \text{ with } (A, B) \mapsto (A \cap H, B \cap N)$$

*is a surjective complete lattice homomorphism.*

A subcontext $(H, N, I \cap (H \times N))$ of a purified context $\mathbb{K}$ is called **arrow–closed** if for $h \in H$, the relation $h \swarrow m$ implies $m \in N$ and for $n \in N$, the relation $g \nearrow n$ implies $g \in H$.

A topological context is called a **standard topological context** if in addition the following hold:

(R) $\mathbb{K}^{\mathcal{T}}$ is reduced;

(S) $gIm \Rightarrow \exists (A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with $g \in A$ and $m \in B$;

(Q) $(\complement I, (\rho \times \sigma)_{|\complement I})$ is a quasicompact space where $\complement I := (G \times M) \setminus I$ and $\rho \times \sigma$ denotes the product topology on $G \times M$.

Let now $L$ be a 0–1 lattice. A nonempty lattice filter $F$ of $L$ is called an **I-maximal filter** [Ur78] if there exists a nonempty lattice ideal $I$ of $L$ such that $F \cap I = \emptyset$ and every proper superfilter $\tilde{F} \supset F$ already contains an element of $I$. We denote the set of all I-maximal proper filters of $L$ by $\mathcal{F}_0(L)$. Dually, the set $\mathcal{I}_0(L)$ of all F-maximal ideals is introduced. The dual space of $L$, called the standard topological context of $L$ is defined by

$$\mathbb{K}^{\mathcal{T}}(L) := ((\mathcal{F}_0(L), \rho_0), (\mathcal{I}_0(L), \sigma_0), \Delta)$$

where $F \Delta I :\Leftrightarrow F \cap I \neq \emptyset$ and the topologies $\rho_0$ and $\sigma_0$ are given by the subbasis

$$\mathcal{S}_{\rho_0} := \{F_a |\ a \in L\}; F_a := \{F \in \mathcal{F}_0(L) |\ a \in F\},$$

$$\mathcal{S}_{\sigma_0} := \{I_a |\ a \in L\}; I_a := \{I \in \mathcal{I}_0(L) |\ a \in I\}.$$

$\mathbb{K}^{\mathcal{T}}(L)$ is the reduced context of all filters and ideals of $L$ and it is a standard topological context. The 0–1 lattice $L$ is isomorphic to $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$ via the isomorphism $\iota_A : L \to \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)); \quad \iota_A(a) = (F_a, I_a)$.

Conversely, every standard topological context $\mathbb{K}^{\mathcal{T}}$ is isomorphic to $\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ via the pair of homeomorphisms

$$\psi_{\mathbb{K}^{\mathcal{T}}} : G \to \mathcal{F}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})),\ g \mapsto \{(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) | g \in A\},$$

$$\phi_{\mathbb{K}^{\mathcal{T}}} : M \to \mathcal{I}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})),\ m \mapsto \{(A,B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) | m \in B\}.$$

Let $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ be standard topological contexts. A pair of maps $(\alpha, \beta)$ with $\alpha : G_1 \to G_2$ and $\beta : M_1 \to M_2$ is called a **context embedding of** $\mathbb{K}_1^{\mathcal{T}}$ **into** $\mathbb{K}_2^{\mathcal{T}}$ if the contexts $\mathbb{K}_1^{\mathcal{T}}$ and $((\alpha(G_1), \rho_{2|\alpha(G_1)}), (\beta(M_1), \sigma_{2|\beta(M_1)}), I_2 \cap (\alpha(G_1) \times \beta(M_1)))$ are isomorphic as topological contexts with respect to $(\alpha, \beta)$.

If $\mathbb{K}^{\mathcal{T}}$ is a topological context, a subcontext $((H, \rho_{|H}), N, \sigma_{|N}), I \cap H \times N))$ is called **weakly compatible** if

$$(A,B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \Rightarrow (A \cap H, B \cap N) \in \underline{\mathcal{B}}(H, N, I \cap (H \times N)).$$

A context embedding $(\alpha, \beta)$ between two standard topological contexts $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ is called a **standard embedding of** $\mathbb{K}_1^{\mathcal{T}}$ **into** $\mathbb{K}_2^{\mathcal{T}}$ if the following conditions are satisfied:

(a) $((\alpha(G_1), \rho_{2|\alpha(G_1)}), (\beta(M_1), \sigma_{2|\beta(M_1)}), I_2 \cap (\alpha(G_1) \times \beta(M_1)))$ is a weakly compatible subcontext of $\mathbb{K}_2^{\mathcal{T}}$;

(b) For $(A,B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$, there exists $(C,D) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$ such that

$$(\alpha(A), \beta(B)) = ((C \cap \alpha(G_1)), (D \cap \beta(M_1))).$$

Preimages of I-maximal filters (resp. ideals) are not maximal again, so we have to define appropriate morphisms between standard topological contexts to improve a categorical dual equivalence between the category of bounded lattices and the category of standard topological contexts.

A multivalued function $F : X \to Y$ from a set $X$ to a set $Y$ is a binary relation $F \subseteq X \times Y$ such that $pr_X(F) = X$, where $pr_X$ denotes the projection onto $X$. For $A \subseteq X$ and $B \subseteq Y$ we define

$$FA := pr_Y(F \cap (A \times Y)) = \{y \in Y \mid (a,y) \in F \text{ for some } a \in A\};$$
$$F^{-1}B := pr_X(F \cap (X \times B)) = \{x \in X \mid (x,b) \in F \text{ for some } b \in B\};$$
$$F^{[-1]}B := \{x \in X \mid Fx \subseteq B\}.$$

Note that $FA = \bigcup_{a \in A} Fa$ and $F^{-1}B = \bigcup_{b \in B} F^{-1}b$. If $F : X \to Y$ and $G : Y \to Z$ are multivalued functions their relational product

$$G \circ F := \{(x,z) \in X \times Z \mid (x,y) \in F \text{ and } (y,z) \in G \text{ for some } y \in Y\}$$

is a multivalued function from $X$ to $Z$.

We shall call a **multivalued standard morphism** from $\mathbb{K}_1^{\mathcal{T}}$ to $\mathbb{K}_2^{\mathcal{T}}$ a pair $(R, S) : \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$, where $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ are standard topological contexts, $R$ is a multivalued function from $G_1$ to $G_2$ and $S$ is a multivalued function from $M_1$ to $M_2$ satisfying the following conditions:

(i) $(R^{[-1]}A, S^{[-1]}B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ for every $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$;

(ii) $Rg = Rg'' = \overline{Rg}$ for every $g \in G_1$ and
$Sm = Sm'' = \overline{Sm}$ for every $m \in M_1$.

**Remark 1.** Condition (ii) can be understood in lattice theoretical terms. Every element $g \in G_1$ correspond to exactly one $I$-maximal filter of $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$. The demand on $Rg$ to be a closed extent means that $Rg$ corresponds to a lattice filter of $\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$.

Every multivalued standard morphism induces a 0-1 lattice homomorphism and vice versa. In order to make this assignment functorial we have to modify the relational composition of multivalued standard morphisms, since the relational composition of two multivalued standard morphisms is not necessarily a multivalued standard morphism.

Let $(R_1, S_1) : \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$ and $(R_2, S_2) : \mathbb{K}_2^{\mathcal{T}} \to \mathbb{K}_3^{\mathcal{T}}$ be multivalued standard morphisms between standard topological contexts. We define

$$(R_2, S_2)\Box(R_1, S_1) := (R_2\Box R_1, S_2\Box S_1)$$

where

$$(R_2\Box R_1)g_1 := ((R_2 \circ R_1)g_1)'' \text{ and } (S_2\Box S_1)m_1 := ((S_2 \circ S_1)m_1)''$$

and $\circ$ denotes the relational product, i.e.

$$(R_2 \circ R_1)g_1 := \{g_3 \in G_3 \mid g_3 \in R_2 g_2 \text{ for some } g_2 \in R_1 g_1\} \text{ and, dually,}$$

$$(S_2 \circ S_1)g_1 := \{m_3 \in M_3 \mid m_3 \in S_2 m_2 \text{ for some } m_2 \in S_1 m_1\}.$$

The class of all standard topological contexts together with the multivalued standard morphisms with $\Box$ as composition yields a category which is dually equivalent to the category of 0-1 lattices with 0-1 lattice homomorphisms.

## 3. Standard Topological Contexts with Pseudometric

If we want to represent several classes of ordered algebraic structures, standard topological contexts are the best tool to do this. On the other hand, if we want to approximate objects by their attributes in a given formal context, we have to modify this approach towards a topological formal concept analysis and to investigate a generalization on formal contexts of the classical notion of a metric (see [Sa00b]).

**Definition 3.1.** Let $G$ and $M$ be two sets. We call **pseudometric between** $G$ and $M$ a map $d : G \times M \to \mathbb{R}$ satisfying the following rectangle condition:

$$(R) \qquad d(g,m) \leq d(g,n) + d(h,m) + d(h,n),\ \ g,h \in G,\ \ m,n \in M,$$

and, for every $g \in G$ and $\varepsilon > 0$, there is an attribute $m \in M$ with $d(g,m) < \varepsilon$. Dually, for every $m \in M$ and every $\varepsilon > 0$, there is an object $g \in G$ with $d(g,m) < \varepsilon$.

If $d$ is a pseudometric between $G$ and $M$, then $d^\vee : G \times G \to \mathbb{R}$ defined by $d^\vee(g,h) := \inf_{m \in M}(d(g,m) + d(h,m)), g,h \in G$ is a pseudometric on $G$. Dually, $d^\wedge : M \times M \to \mathbb{R}$ defined by $d^\wedge(m,n) := \inf_{g \in G}(d(g,m) + d(g,n)), m,n \in M$ is a pseudometric on $M$.

**Definition 3.2.** A formal context $\mathbb{K} := (G,M,I)$ is called a **pseudometric context** if there is a pseudometric $d : G \times M \to \mathbb{R}$ between $G$ and $M$ satisfying the following two conditions, called $\varepsilon$-conditions:

$$\forall \varepsilon \geq 0\ \forall g \in G\ \exists m \in M :\ gIm \text{ and } d(g,m) < \varepsilon,$$

$$\forall \varepsilon \geq 0\ \forall m \in M\ \exists g \in G :\ gIm \text{ and } d(g,m) < \varepsilon.$$

We shall call a pseudometric context **standard** if $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\} = 0$ for every concept $(A,B)$ of $\mathbb{K}$.

Let $\mathbb{K} := (G,M,I;d)$ be a pseudometric context. We consider $G$ and $M$ as topological spaces with the pseudometric topology given by $d^\vee$ and $d^\wedge$, respectively. As we have seen before, a topological context is a triple $(G,M,I)$ where $G$ and $M$ are topological spaces and $I \subseteq G \times M$ is a binary relation between them, satisfying some compatibility conditions with the topologies on $G$ and $M$. If in addition the topological context satisfies some separation and compactness properties, it is called a standard topological context and it was shown by G. Hartung that the category of standard topological contexts is dual equivalent to that of 0–1 lattices.

A question arises naturally: what are the connections between pseudometric contexts and standard topological ones? As in the topological algebra, there are two possibilities. On the one hand, we can demand that the topologies on $G$ and $M$ are generated by the pseudometrics induced by $d$. We shall call such a context a **compatible pseudometric context**, i.e., $\mathbb{K}^{\mathcal{T}} := ((G, \mathcal{T}_{d^\vee}), (M, \mathcal{T}_{d^\wedge}, I)$ is a standard topological context where $\mathcal{T}_{d^\vee}$ denotes the pseudometric topology on $G$, and $\mathcal{T}_{d^\wedge}$ denotes the pseudometric topology on $M$.

On the other hand, we can simply consider **standard topological contexts with pseudometric**, i.e., no compatibility conditions between the topologies on $G$ and $M$ and the given pseudometric are required.

In the following we shall investigate the categories of standard topological contexts with a compatible pseudometric or not and we shall take a look whether an extension of the Hartung duality to the pseudometric case is possible or not. Beside of this extension, we are mainly interested on how some properties of pseudometric lattice homomorphisms are reflected into the properties of standard topological context morphisms.

**Remark 2.** Before starting these investigations, remember that $\overline{\mathbb{R}}$ can be understood as the concept lattice of the context $(\mathbb{Q}, \mathbb{Q}, \leq)$. Since $(\mathbb{Q}, d)$ is a metric space, where $d$ is the natural metric on $\mathbb{Q}$, then $(\mathbb{Q}, \mathbb{Q}, \leq)$ is a metric context. The metric on $\overline{\mathbb{R}} \simeq \underline{\mathcal{B}}(\mathbb{Q}, \mathbb{Q}, \leq)$ can be understood as a kind of "reflection" of the contextual metric $d$ on $(\mathbb{Q}, \mathbb{Q}, \leq)$ on the concept lattice. The following Lemma ([Sa00b]) synthesizes this phenomenon in its full generality.

**Lemma 3.1.** *Let $\mathbb{K} := (G, M, I; \rho)$ be a pseudometric context. The map $d : \underline{\mathcal{B}}(G, M, I) \times \underline{\mathcal{B}}(G, M, I) \to \mathbb{R}$, defined by*

$$d((A, B), (C, D)) := \max\{\rho(A, D), \rho(C, B)\},$$

*is a pseudometric on $\underline{\mathcal{B}}(G, M, I)$, the concept lattice of $\mathbb{K}$.*

Let $\mathbb{K}_1 := (G_1, M_1, I_1; d_1)$ and $\mathbb{K}_2 := (G_2, M_2, I_2; d_2)$ be standard topological contexts with pseudometric. A morphism between them is defined as a pair of multivalued functions $R : G_1 \to G_2$ and $S : M_1 \to M_2$ (i.e., $R$ and $S$ are binary relations on $G_1 \times G_2$ and $M_1 \times M_2$, respectively, satisfying $\mathrm{pr}_{G_1} R = G_1$ and $\mathrm{pr}_{M_1} S = M_1$) with the properties:

(i) $(R^{[-1]}A, S^{[-1]}B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ for every $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$;

(ii) $Rg = Rg'' = \overline{Rg}$ for every $g \in G_1$, and

$Sm = Sm'' = \overline{Sm}$ for every $m \in M_1$;

(iii) $d_2(Rg, Sm) \leq d_1(g, m)$ for every $g \in G_1$ and $m \in M_1$.

The pair $(R, S)$ will be called multivalued pseudometric morphism.

**Lemma 3.2.** *The class of all standard topological contexts with pseudometric together with the multivalued pseudometric morphisms between them yields a category denoted by* $\mathbf{T}opCon_d$. *The class of all compatible pseudometric contexts is a full subcategory* **CCM** *of* $\mathbf{T}opCon_d$.

**Proof.** Let $\mathbb{K} := (G, M, I; d)$ be a pseudometric context, the identity $(R_e, S_e)$ where $R_e : G \to G$ and $S_e : M \to M$ are defined by $R_e g := g''$ and $S_e m := m''$, respectively, is a multivalued standard morphism. Since $d(R_e g, S_e m) = d(g'', m'') \leq d(g, m)$ for every $g \in G$ and $m \in M$, we conclude that $(R_e, S_e)$ is a multivalued pseudometric morphism, i.e., the identity in the category of standard topological contexts is also identity in $\mathbf{T}opCon_d$.

Let now $(R_1, S_1) : (\mathbb{K}_1, d_1) \to (\mathbb{K}_2, d_2)$ and $(R_2, S_2) : (\mathbb{K}_2, d_2) \to (\mathbb{K}_3, d_3)$ be multivalued metric morphisms. We shall prove that their composition $(R_2, S_2)\Box(R_1, S_1) := (R_2\Box R_1, S_2\Box S_1)$ is again a multivalued pseudometric morphism.

By definition of $\Box$, $(R_2\Box R_1)g_1 := ((R_2 \circ R_1)g_1)''$ for every $g_1 \in G_1$. Dually, we have $(S_2\Box S_1)m_1 := ((S_2 \circ S_1)m_1)''$ for every $m_2 \in M_2$. The following holds:

$$\begin{aligned} d_3((R_2\Box R_1)g_1, (S_2\Box S_1)m_1) &= d_3(((R_2 \circ R_1)g_1)'', ((S_2 \circ S_1)m_1)'') \\ &\leq d_3((R_2 \circ R_1)g_1, (S_2 \circ S_1)m_1) \\ &= d_3(R_2(R_1 g_1), S_2(S_1 m_1)) \\ &= d_3(\{g_3 \in G_3 \mid g_3 \in R_2 g_2 \text{ for some } g_2 \in R_1 g_1\}, \\ &\qquad \{m_3 \in M_3 \mid m_3 \in R_2 m_2 \text{ for some } m_2 \in R_1 m_1\}). \end{aligned}$$

But $(R_1, S_1)$ and $(R_2, S_2)$ are multivalued pseudometric morphisms, and so

$$d_2(R_1 g_1, S_1 m_1) \leq d_1(g_1, m_1) \text{ for every } g_1 \in G_1 \text{ and } m_1 \in M_1$$

$$d_3(R_2 g_2, S_2 m_2) \leq d_2(g_2, m_2) \text{ for every } g_2 \in G_2 \text{ and } m_2 \in M_2.$$

For every $g_2 \in R_1 g_1$ and $m_2 \in S_1 m_1$, we have $d_3(R_2 g_2, S_2 m_2) \leq d_2(g_2, m_2)$ which implies

$$\begin{aligned}
d_3(R_2(R_1 g_1), S_2(S_1 m_1)) &= d_3\Big(\bigcup_{g_2 \in R_1 g_1} R_2 g_2, \bigcup_{m_2 \in S_1 m_1} S_2 m_2\Big) \\
&= \inf\{d_3(g_3, m_3) \mid g_3 \in \bigcup_{g_2 \in R_1 g_1} R_2 g_2, \\
&\qquad m_3 \in \bigcup_{m_2 \in S_1 m_1} S_2 m_2\} \\
&\leq \inf\{d_3(g_3, m_3) \mid g_3 \in R_2 g_2, m_3 \in S_2 m_2\}, \\
&\qquad \text{for every } g_2 \in R_1 g_1 \text{ and every } m_2 \in S_1 m_1 \\
&= d_3(R_2 g_2, S_2 m_2) \text{ for every } g_2 \in R_1 g_1, m_2 \in S_1 m_1 \\
&\leq d_2(g_2, m_2) \text{ for every } g_2 \in R_1 g_1, m_2 \in S_1 m_1.
\end{aligned}$$

Hence $d_3(R_2(R_1 g_1), S_2(S_1 m_1)) \leq \inf\{d_2(g_2, m_2) \mid g_2 \in R_1 g_1, m_2 \in S_1 m_1\} = d_2(R_1 g_1, S_1 m_1) \leq d_1(g_1, m_1)$. Since associativity is naturally inherited, the above condition completes our proof. □

**Lemma 3.3.** *If $(L, \rho)$ is a 0-1-lattice and $\rho : L \times L \to \mathbb{R}$ is a pseudometric on $L$, then $\mathbb{K}^{\mathcal{T}}(L)$, the standard topological context of $L$, is a standard pseudometric context.*

**Proof.** As we have seen before, to every 0-1-lattice $L$, we can define a standard topological context denoted by $\mathbb{K}^{\mathcal{T}}(L) := (\mathcal{F}_0(L), \mathcal{I}_0(L), \Delta)$ where $F \Delta I :\Leftrightarrow F \cap I \neq \emptyset$.

We shall define a pseudometric $d : \mathcal{F}_0(L) \times \mathcal{I}_0(L) \to \mathbb{R}$ on $\mathbb{K}^{\mathcal{T}}(L)$, by $d(F, I) := \inf\{\rho(g, m) \mid g \in F, m \in I\} = \rho(F, I)$. Let $F \in \mathcal{F}_0(L)$. Then $d(F, F') = d(F, \{I \in \mathcal{I}_0(L) \mid F \cap I \neq \emptyset\}) = 0$ since $d(F, I) = 0$ for every $I \in \mathcal{I}_0(L)$ with $F \cap I \neq \emptyset$, i.e., $I \in F'$. Let us prove the rectangle inequality for $d$. Let $(F, I), (F, J), (K, J)$ and $(K, I)$ in $\mathcal{F}_0(L) \times \mathcal{I}_0(L)$ be arbitrary chosen. We have to prove that

$$d(F, I) \leq d(F, J) + d(K, J) + d(K, I).$$

Then

$$\begin{aligned}
d(F,I) &= \inf\{\rho(f,i) \mid f \in F, i \in I\} \\
&\le \inf\{\rho(f,i) + \rho(k,j) + \rho(k,i) \mid f \in F, i \in I\} \text{ for } j \in J, k \in K \\
&\le \inf\{\rho(f,i) + \rho(k,j) + \rho(k,i) \mid f \in F, i \in I, j \in J, k \in K\} \\
&\le \inf\{\rho(f,i) \mid f \in F, i \in I\} + \inf\{\rho(k,j) \mid k \in K, j \in J\} \\
&+ \inf\{\rho(k,i) \mid k \in K, i \in I\} \\
&= d(F,J) + d(K,J) + d(K,I).
\end{aligned}$$

If $(A,B) \in \underline{\mathcal{B}}(\mathbb{K}^{\mathcal{T}}(L))$, we conclude that $d(A,B) = 0$ by the definition of the incidence relation and of the set distance; hence $(\mathbb{K}^{\mathcal{T}}(L), d)$ is a standard pseudometric context. □

**Remark 3.** Since $d^{\vee}$ is the pseudometric induced on $\mathcal{F}_0(L)$ by $d$, we have $d^{\vee}(F_1,F_2) = \inf\{d(F_1,I) + d(F_2,I) \mid I \in \mathcal{I}_0\}$; hence we conclude that generally, the pseudometric $d^{\vee}$ does not induce the topology on $\mathcal{F}_0(L)$ (which has as subbasis of closed sets the family $\{F_a \mid a \in L\}$, where $F_a := \{F \in \mathcal{F}_0(L) \mid a \in F\}$). Indeed, for two filters $F_1, F_2 \in \mathcal{F}_0(L)$ we will often be able to find an ideal $I \in \mathcal{I}_0(L)$ which has a non empty intersection to $F_1$ and $F_2$ and therefore $d^{\vee}(F_1,F_2) = 0$.

In the following we shall consider only the case where $\mathbb{K}^{\mathcal{T}}$ is a standard topological context with pseudometric. Let $\mathbb{K}^{\mathcal{T}} := (G,M,I)$ be a standard topological context and let $(P_\varepsilon)_{\varepsilon \ge 0}$ be a family of non empty relations, $P_\varepsilon \subseteq G \times M$ with $\varepsilon \ge 0$, which are satisfying the following conditions:

$(M')$ $\qquad P_\varepsilon(x,y) \to P_\delta(x,y),\ \delta \ge \varepsilon$

$$P_\varepsilon \wedge P_\delta(k,z) \wedge P_\eta(k,y) \to P_{\varepsilon+\delta+\eta}(x,y).$$

$(M_\infty)$ $\qquad \forall \delta \ge \varepsilon : P_\delta(x,y) \to P_\varepsilon(x,y), \varepsilon \ge 0.$

$(M_0)$ $\qquad \forall g\ \forall \varepsilon\ \exists m : P_\varepsilon(x,y).$

A morphism $(R,S) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \ge 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon \ge 0}$ has to satisfy the following compatibility condition

$(C)$ $\qquad P_\varepsilon(g_1,m_1) \Rightarrow \exists g_2 \in Rg_1\ \exists m_2 \in Sm_1 : Q_\varepsilon(g_2,m_2), \varepsilon \ge 0.$

**Lemma 3.4.** *The class of multicontexts* $(\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0}$*, where* $\mathbb{K}^{\mathcal{T}}$ *is a standard topological context and* $(P_\varepsilon)_{\varepsilon\geq 0}$ *a family of binary relations on* $G \times M$ *satisfying* $(M')$*,* $(M_\infty)$ *and* $(M_0)$*, together with the multivalued standard morphisms which are satisfying condition* $(C)$ *yields a category denoted by* $\mathbf{TopCon}_\varepsilon$*.*

**Proof.** Let $\varepsilon \geq 0$ be arbitrary chosen and $\mathbb{K}^{\mathcal{T}}$ be a standard topological context. The identity morphism $(R_\varepsilon, S_\varepsilon) : \mathbb{K}^{\mathcal{T}} \to \mathbb{K}^{\mathcal{T}}$ where $R_e g := g''$ and $S_e m := m''$ is obviously satisfying condition $(C)$. Let us now consider $(R_1, S_1) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0}$ and $(R_2, S_2) : (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_3^{\mathcal{T}}, R_\varepsilon)_{\varepsilon\geq 0}$ two morphisms between objects in $\mathbf{TopCon}_\varepsilon$. We shall prove that their composition in **Topcon**, i.e., $(R_2, S_2)\Box(R_1, S_1) = (R_2\Box R_1, S_2\Box S_1)$ is a morphism between objects of $\mathbf{TopCon}_\varepsilon$, i.e.,

$$P_\varepsilon(g_1, m_1) \Rightarrow \exists g_3 \in (R_2\Box R_1)g_1 \ \exists m_3 \in (S_2\Box S_1)m : R_\varepsilon(g_3, m_3).$$

Since the given morphisms are satisfying condition $(C)$ and, by definition, $(R_2\Box R_1)g_1 := ((R_2 \circ R_1)g_1)''$ and $(S_2\Box S_1)m_1 = ((S_2\Box S_1)m_1)''$, we conclude that $P_\varepsilon(g_1, m_1)$ implies the existence of a $g_2 \in R_1 g_1$ and an $m_2 \in S_1 m_1$ with $Q_\varepsilon(g_2, m_2)$, which implies the existence of elements $g_3 \in R_2(R_1 g_1)$ and $m_3 \in S_2(S_1 m_1)$ with $R_\varepsilon(g_3, m_3)$. Since $g_3 \in R_2(R_1 g_1) \subseteq (R_2 \circ R_1)g_1''$ and $m_3 \in S_2(S_1 m_1) \subseteq (S_2 \circ S_1)m_1''$ our proof is complete. □

**Proposition 3.5.** *The category* $\mathbf{TopCon_d}$ *of standard topological contexts with pseudometric is equivalent to* $\mathbf{TopCon}_\varepsilon$*.*

**Proof.** Let $F : \mathbf{TopCon_d} \to \mathbf{TopCon}_\varepsilon$ defined on objects by $F(\mathbb{K}^{\mathcal{T}}, d) = (\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0}$ and on morphisms in an obvious way. The functor $F$ is obviously faithful. Let $(R, S) : F(\mathbb{K}_1^{\mathcal{T}}, d_1) \to F(\mathbb{K}_2^{\mathcal{T}}, d_2)$ be a morphism of $\mathbf{TopCon}_\varepsilon$, that means $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0}$. We only have to prove that $d_2(Rg_1, Sm_1) \leq d_1(g_1, m_1)$ for every $g_1 \in G_1$ and $m_1 \in M_1$.

Let $g_1 \in G_1$ and $m_1 \in M_1$ be arbitrary chosen and define $\varepsilon := d_1(g_1, m_1)$. It follows that $P_\varepsilon(g_1, m_1)$ and by $(C)$, there is a $g_2 \in Rg_1$ and an $m_2 \in Sm_1$ with $Q_\varepsilon(g_2, m_2)$, i.e., $d_2(g_2, m_2) \leq \varepsilon$. Hence $d_2(Rg_1, Sm_1) = \inf\{d_2(g_2, m_2) \mid g_2 \in Rg_1, m_2 \in Sm_1\} \leq \varepsilon$, i.e., $F$ is full.

If $(\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0}$ is an object in $\mathbf{TopCon}_\varepsilon$, we define a pseudometric $d : G \times M \to [0, +\infty]$ by $d(g, m) := \inf\{\delta \geq 0 \mid P_\delta(g, m)\}$. As we have seen in the precedent section, $d$ is well-defined and is a pseudometric between $G$ and $M$. Hence, we have found an object $(\mathbb{K}^{\mathcal{T}}, d)$ in $\mathbf{TopCon_d}$ with $F(\mathbb{K}^{\mathcal{T}}, d) \cong (\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \geq 0}$ which concludes the proof. □

We shall now investigate whether the duality between standard topological contexts and 0–1 lattices can be extended to the metric case. Even this will not be generally true, it is of interest to investigate how some properties of morphisms between pseudometric 0–1 lattices are reflected in the category $\mathbf{T}opCon_d$ as equivalent properties of standard multivalued morphisms.

Let $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ be a morphism in $\mathbf{TopCon_d}$. This morphism induces $f_{RS} : (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}), \rho_2) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}), \rho_1)$ a 0–1 lattice homomorphism defined by $f_{RS}(A, B) = (R^{[-1]}A, S^{[-1]}B)$ where $R^{[-1]}A = \{g_1 \in G_1 \mid Rg_1 \subseteq A\}$ and $S^{[-1]}B = \{m_1 \in M_1 \mid Sm_1 \subseteq D\}$. By definition,

$$\begin{aligned} \rho_1(f_{RS}(A, B), f_{RS}(C, D)) &= \rho_1((R^{[-1]A}, S^{[-1]}B), (R^{[-1]}C, S^{[-1]}D) \\ &= \max\{d_1(R^{[-1]}A, S^{[-1]}D), d_1(R^{[-1]}C, S^{[-1]}B)\}. \end{aligned}$$

The morphism $(R, S)$ is in $\mathbf{TopCon_d}$, i.e., it satisfies $d_2(Rg_1, Sm_1) \leq d_1(g_1, m_1)$ for every $g_1 \in G_1$ and $m_1 \in M_1$; hence $d_1(g_1, m_1) \geq d_2(Rg_1, Sm_1) \geq d_2(A, D)$ for every $g_1 \in R^{[-1]}A$ and every $m \in S^{[-1]}D$, and so $d_1(R^{[-1]}A, S^{[-1]}D) \geq d_2(A, D)$. By a similar calculus, we obtain $d_1(R^{[-1]}C, S^{[-1]}B) \geq d_2(C, B)$ which implies the following inequality:

$$\rho_1(f_{RS}(A, B), f_{RS}(C, D)) \geq \rho_2((A, B), (C, D)).$$

As we can see, condition $(iii)$ has as consequence that on the "lattice side" the mappings are not the usually contractions. To avoid this, we will impose for context morphisms the following compatibility condition

$(iv)$ $\quad d_1(R^{-1}g_2, S^{-1}m_2) \leq d_2(g_2, m_2)$ for every $g_2 \in G_2$ and $m_2 \in M_2$.

In fact, let $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ be such a morphism and consider $f_{RS} : (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}), \rho_2) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}), \rho_1)$ the induced 0–1 lattice homomorphism. Then

$$\rho_1(f_{RS}(A, B), f_{RS}(C, D)) = \max\{d_1(R^{[-1]}A, S^{[-1]}D), d_1(R^{[-1]}C, S^{[-1]}B)\}.$$

Since $R^{[-1]}A = \{g_1 \in G_1 \mid Rg_1 \subseteq A\}$ where $A \subseteq G_2$, we have $R^{[-1]}A = \bigcup_{T\subseteq A, T\neq\emptyset} \bigcap_{a\in T} R^{-1}a$ and $S^{[-1]}D = \bigcup_{T'\subseteq D, T'\neq\emptyset} \bigcap_{d\in T'} S^{-1}d$. It follows

$$\begin{aligned} d_1(R^{[-1]}A, S^{[-1]}D) &= d_1(\bigcup_{T\subseteq A, T\neq\emptyset} \bigcap_{a\in T} R^{-1}a, \bigcup_{T'\subseteq D, T'\neq\emptyset} \bigcap_{d\in T'} S^{-1}d) \\ &\leq d_1(\bigcap_{a\in T} R^{-1}a, \bigcap_{d\in T'} S^{-1}d) \end{aligned}$$

for every non empty subsets $T \subseteq A$ and $T' \subseteq D$. Choose $T := \{a\}$ with $a \in A$ and $T' := \{d\}$ with $d \in D$, then

$$d_1(R^{[-1]}A, S^{[-1]}D) \leq d_1(R^{-1}a, S^{-1}d) \leq d_2(a, d)$$

for every $a \in A$ and $d \in D$; hence $d_1(R^{[-1]}A, S^{[-1]}D) \leq d_2(A, D)$.

Analogously, we are able to prove that $d_1(R^{[-1]}C, S^{[-1]}B) \leq d_2(C, B)$; hence $f_{RS}$ is a contraction.

**Remark 4.** The dual inequality to (iv), i.e.,

$$d_1(R^{-1}g_2, S^{-1}m_2) \geq d_2(g_2, m_2)$$

implies (iii). Indeed, for every $g_1 \in G_1$ and $m_1 \in M_1$, we have

$$d_2(Rg_1, Sm_1) = \inf d_2(g_2, m_2) \leq \inf d_1(R^{-1}g_2, S^{-1}m_2) \leq d_1(g_1, m_1).$$

**Lemma 3.6.** *The class of standard topological contexts with metric together with all multivalued standard morphisms between them satisfying condition* (*iv*) *yields a category denoted by* $\mathbf{TopCon'_d}$.

**Proof.** The unit morphism $(R_e, S_e) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}, d)$ defined by $R_e g := g''$ and $S_e m := m''$ satisfies $(C')$, since $R_e^{-1}h = \{g \in G \mid h \in g''\}$ and $S_e^{-1}n = \{m \in M \mid n \in m''\}$. In particular, $h \in R^{-1}h$ and $n \in S^{-1}n$, hence $d(R_e^{-1}h, S_e^{-1}n) \leq d(h, n)$ for every $h \in G$ and $n \in M$.

Let $(R_1, S_1) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ and $(R_2, S_2) : (\mathbb{K}_2^{\mathcal{T}}, d_2) \to (\mathbb{K}_3^{\mathcal{T}}, d_3)$ be two multivalued standard morphisms which are satisfying $(iv)$. We shall prove that their composition $(R_2, S_2)\Box(R_1, S_1) = (R_2\Box R_1, S_2\Box S_1) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ is also satisfying $(iv)$.

For $g_3 \in G_3$ and $m_3 \in M_3$ we have

$$\begin{aligned}(R_2 \Box R_1)^{-1} g_3 &= \{g_1 \in G_1 \mid g_3 \in ((R_2 \circ R_1) g_1)''\} \\ &\supseteq \{g_1 \in G_1 \mid g_3 \in R_2(R_1 g_1)\} \\ &= \{g_1 \in G_1 \mid g_1 \in (R_2 \circ R_1)^{-1} g_3\} \\ &= \{g_1 \in G_1 \mid g_1 \in R_1^{-1}(R_2^{-1} g_3))\} \\ &= \{g_1 \in G_1 \mid \exists g_2 \in G_2 : (g_2, g_1) \in R^{-1}, (g_3, g_2) \in R_2^{-1}\} \\ &= \{g_1 \in G_1 \mid \exists g_2 \in G_2 : \ g_1 \in R_1^{-1} g_2, g_2 \in R_2^{-1} g_3\}.\end{aligned}$$

In a similar manner, we are able to prove that

$$(S_2 \Box S_1)^{-1} m_3 \supseteq \{m_1 \in M_1 \mid \exists m_2 \in M_2 : \ m_1 \in S^{-1} m_2, m_2 \in S_2^{-1} m_3\},$$

hence

$$\begin{aligned}&d_1((R_2 \Box R_1)^{-1} g_3, (S_2 \Box S_1)^{-1} m_3) \\ &\quad \leq d_1(\{g_1 \in G_1 \mid \exists g_2 \in G_2 : \ g_1 \in R_1^{-1} g_2, \ g_2 \in R_2^{-1} g_3\}, \\ &\qquad \{m_1 \in M_1 \mid \exists m_2 \in M_2 : \ m_1 \in S_1^{-1} m_2, \ m_2 \in S_2^{-1} m_2\}) \\ &\quad = \inf\{d_1(g_1, m_1) \mid \exists g_2 \in G_2 : \ g_1 \in R_1^{-1} g_2, \ g_2 \in R_2^{-1} g_3, \\ &\qquad \exists m_2 \in M_2 : \ m_1 \in S_1^{-1} m_2, m_2 \in S_2^{-1} m_3\} \\ &\quad = \inf\{d_1(R_1^{-1} g_2, S_2^{-1} m_2) \mid g_2 \in R_2^{-1} g_3, m_2 \in S_2^{-1} m_3\} \\ &\quad \leq \inf\{d_2(g_2, m_2) \mid g_2 \in R_2^{-1} g_3, m_2 \in S_2^{-1} m_3\} \\ &\quad = d_2(R_2^{-1} g_3, S_2^{-1} m_3) \leq d_3(g_3, m_3).\end{aligned}$$

□

If we split again the pseudometric $d : G \times M \to [0, +\infty]$ in the family of relations $(P_\varepsilon)_{\varepsilon \geq 0}$, compatibility condition $(C')$ for $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon \geq 0}$ changes to

$$(C') \qquad Q_\varepsilon(g_2, m_2) \Rightarrow \exists g_1 \in R^{-1} g_2 \ \exists m_1 \in S^{-1} m_2 : P_\varepsilon(g_1, m_1).$$

**Lemma 3.7.** *The class of multicontexts $(\mathbb{K}^{\mathcal{T}}, P_\varepsilon)_{\varepsilon \geq 0}$, where $\mathbb{K}^{\mathcal{T}}$ is a standard topological context and $P_\varepsilon$ is a binary relation between the object and the attribute* *set of $\mathbb{K}^{\mathcal{T}}$ satisfying axioms $(M')$, $(M_0)$ and $(M_\infty)$ is the object class of a category denoted* $\mathbf{TopCon}'_\varepsilon$*, whose morphisms are the multivalued standard morphisms which are satisfying $(C')$.*

**Proof.** The identity $(R_e, S_e)$ is obviously a morphism in $\mathbf{TopCon}'_\varepsilon$. Let $(R_1, S_1) : (\mathbb{K}_1^{\mathcal{T}}, P_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0}$ and $(R_2, S_2) : (\mathbb{K}_2^{\mathcal{T}}, Q_\varepsilon)_{\varepsilon\geq 0} \to (\mathbb{K}_3^{\mathcal{T}}, R_\varepsilon)_{\varepsilon\geq 0}$ be morphisms in $\mathbf{TopCon}'_\varepsilon$. Their composition is again in $\mathbf{TopCon}'_\varepsilon$. To see this, let $g_3 \in G_3$ and $m_3 \in M_3$ with $R_\varepsilon(g_3, m_3)$. Then there are $g_2 \in R_2^{-1} g_3$ and $m_2 \in S_2^{-1} m_3$ with $Q_\varepsilon(g_2, m_2)$, hence there are $g_1 \in R_1^{-1} g_2$ and $m_1 \in S_1^{-1} m_2$ with $P_\varepsilon(g_1, m_1)$. Now,

$$g_1 \in R_1^{-1} g_2 \subseteq (R_2 \circ R_1)^{-1} g_3 \subseteq (R_2 \Box R_1)^{-1} g_3,$$

$$m_1 \in S_1^{-1} m_2 \subseteq (S_2 \circ S - 1)^{-1} m_3 \subseteq (S_2 \Box S_1)' m_3$$

which completes the proof. □

Proceeding in a similar manner as before, we can prove the following Lemma:

**Lemma 3.8.** *The categories* $\mathbf{TopCon}'_{\mathbf{d}}$ *and* $\mathbf{TopCon}'_\varepsilon$ *are equivalent.*

Let us denote the functors from the Hartung duality by $\mathbf{T}$ and $\mathbf{S}$. The functor $\mathbf{T} : Lat \to TopCon$ is defined on objects by $\mathbf{T}(L) = \mathbb{K}^{\mathcal{T}}(L)$ and for any morphism $f : L_1 \to L_2$, the image of $f$ by $\mathbf{T}$ is a multivalued standard morphism $\mathbf{T}f : \mathbb{K}^{\mathcal{T}}(L_2) \to \mathbb{K}^{\mathcal{T}}(L_1)$ defined by $\mathbf{T}f = (R_f, S_f)$ where

$$R_f \subseteq \mathcal{F}_0(L_2) \times \mathcal{F}_0(L_1),\ (F_2, F_1) \in R_f \Leftrightarrow f^{-1}(F_2) \subseteq F_1,$$

$$S_f \subseteq \mathcal{I}_0(L_2) \times \mathcal{I}_0(L_1),\ (I_2, I_1) \in S_f \Leftrightarrow f^{-1}(I_2) \subseteq I_1.$$

The functor $\mathbf{S}$ is defined on objects by $\mathbf{S}(\mathbb{K}^{\mathcal{T}}) := \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ and for every multivalued standard morphism $(R, S) : \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$, the image of $(R, S)$ by $\mathbf{S}$ is a 0-1-lattice homomorphism $\mathbf{S} : \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}) \to \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ is defined by $\mathbf{S}(R, S) := f_{RS}$ where $f_{RS}(A, B) := (R^{[-1]}A, S^{[-1]}B)$ for all closed concepts $(A, B)$ in $\mathbb{K}_2^{\mathcal{T}}$.

As we have seen before, the restriction of $\mathbf{S}$ to the metric case, $\mathbf{S} : TopCon_d \to Lat_d$, is well-defined, the morphisms in $\mathbf{L}at_d$ being the expansive mappings with respect to the correspondent pseudometric of a lattice $L$ in $\mathbf{L}at_d$.

Consider now $f : (L_1, \rho_1) \to (L_2, \rho_2)$ satisfying $\rho_2(f(x), f(y)) \geq \rho_1(x, y)$ for every $x, y \in L_1$. Then $d_1(F_1, I_1) \leq d_2(f(F_1), f(I_1)) \leq d_2(F_2, I_2)$ for all $F_2 \in R_f^{-1} F_1$ and $I_2 \in S_f^{-1} I_1$ since $(F_2, F_1) \in R_f$ is equivalent to $F_2 \subseteq f(F_1)$, and $(I_2, I_1) \in S_f$ is equivalent to $I_2 \subseteq f(I_1)$. It follows that $d_1(F_1, I_1) \leq d_2(R_f^{-1} F_1, S_f^{-1} I_1)$. Moreover,

$d_1(R_f F_2, S_f I_2) = \inf d_1(F_1, I_1) \leq \inf d_2(f(F_1), f(I_1)) \leq d_2(F_2, I_2)$ which proves that if we consider expansive mappings as morphisms between metric lattices, both (iii) and the dual of (iv) are satisfied, i.e., the restriction of $\mathbf{T}$ to the metric case $\mathbf{T} : Lat_d \to TopCon_d$ is also well-defined.

**Example:**

Let us consider the following lattice the metric being labeled on its Hesse-diagram, the morphism $f$ being given by arrows:

As one can easily check, $f$ is an expansion. Now

$$\begin{aligned} R_f^{-1} &= \{F_2 \in \mathcal{F}_0(L_2) \mid (F_2, F_1) \in R_f\} \\ &= \{F_2 \in \mathcal{F}_0(L_2) \mid f^{-1}(F_2) \subseteq F_1\} \\ &= \{F_2, F_3, [1)\} \end{aligned}$$

and, dually, $S_f^{-1}(I_1) = \{I_2, I_3, (0]\}$. As we can easily see, $d_2(F_2, I_2) = 4, d_1(F_1, I_1) = 3$ i.e., the dual of (iv) (and so (iii)) is satisfied.

**Remark 5.** While dealing with mappings between (pseudo)metric spaces, contractions are more often used as expansive maps. We are considering expansions in this section in order to give a necessary condition that the isomorphisms $\iota : (L, \rho) \to$

$(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), d)$ and $(R_\alpha, S_\beta) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \rho)$ belong to the considered categories.

Unfortunately, the map $\iota : (L, \rho) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), d)$ even if an isomorphism in **Lat**, fails to be an isomorphism in $\mathbf{L}at_d$ (i.e., a bijective isometry). Indeed, $d(\iota a, \iota b) = d((F_a, I_a), (F_b, I_b)) = \max\{\rho(F_a, I_b), \rho(F_b, I_b)\} \leq \rho(a, b)$ for every $a, b \in L$. Obviously, $\iota$ can not generally be an isometry and so the categories $\mathbf{L}at_d$ and $\mathbf{T}opCon_d$ fails to be dual equivalent.

On the other hand, consider $(\mathbb{K}^{\mathcal{T}}, d)$ a standard topological context in $\mathbf{T}opCon_d$. Then $(R_{\alpha_1}, S_{\beta_1}) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \rho)$ is a multivalued pseudometric morphism. Indeed, consider $F \in \mathcal{F}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ and $I \in \mathcal{I}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$. Then for every $g \in R_\alpha^{-1}F$ and $m \in S_\beta^{-1}I$,

$$\rho(F, I) = \rho(\alpha(g)'', \beta(m)'') \leq \rho(\alpha(g), \beta(m)) \leq \sigma((A, B), (C, D))$$

for every $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with $g \in A$, and every $(C, D) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with $m \in D$. Since $\sigma((A, B), (C, D)) = \max\{d(A, D), d(C, B)\}$, choose for $(A, B) := (G, \emptyset)$ and for $(C, D) := (\emptyset, M)$. Then $d(C, B) = 0$ and $d(A, D) \leq d(g, m)$. It follows that $\rho(F, I) \leq d(R_\alpha^{-1}F, S_\beta^{-1}I)$, i.e., the dual of (iv) which then implies (iii).

**Remark 6.** Generally, $(R_\alpha, S_\beta)$ can not be an isomorphism in $\mathbf{T}opCon_d$ since from the above calculus we deduce that $\rho(F, I) = 0$ for every $F \in \mathcal{F}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ and $I \in \mathcal{I}_0(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$, and so **T** and **S** are failing even to be adjoint.

Even if contractions or expansions between pseudometric spaces have proved their usefulness several times, for example in establishing a duality between pseudometric complete lattices and pseudometric contexts, the condition that a map is expansive is too strong in order to obtain a categorical duality or an adjoint situation between the restrictions of the two functors **T** and **S** to the (pseudo)metric case. More general, the most used morphisms between pseudometric spaces are continuous maps. In the following we shall define the analogous concept in the case of standard topological contexts with pseudometric.

**Definition 3.3.** The multivalued standard morphism $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$ between two standard topological contexts with pseudometric is called **pseudometric continuous** if for every $\varepsilon > 0$ and every $g_2 \in G_2$, there is a $\delta > 0$ so that for every $m_2 \in M_2$, with $d_2(g_2, m_2) < \delta$, we have $d_1(R^{-1}g_2, S^{-1}m_2) < \varepsilon$.

The morphism $(R, S)$ is called **pseudometric uniformously continuous** if for every $\varepsilon > 0$, every $g_2 \in G_2$ and $m_2 \in M_2$ there is a $\delta > 0$ such that $d_2(g_2, m_2) < \delta$ implies $d_1(R^{-1}g_2, S^{-1}m_2) < \varepsilon$.

We shall denote the category of standard topological contexts with pseudometric with pseudometric continuous morphisms by $\mathbf{T}C_d$ and that of pseudometric lattices with continuous lattice homomorphism by $\mathbf{L}_d$ and we shall prove that the restrictions of the well-known functors $\mathbf{T}$ and $\mathbf{S}$ of the Hartung duality, $\mathbf{T} : L_d \to TC_d$ and $\mathbf{S} : TC_d \to L_d$, respectively, are well-defined. We have seen before that the object maps of $\mathbf{T}$ and $\mathbf{S}$, respectively, are well-defined.

**Proposition 3.9.** *For every pseudometric continuous standard multivalued morphism* $(R, S) : (\mathbb{K}_1^{\mathcal{T}}, d_1) \to (\mathbb{K}_2^{\mathcal{T}}, d_2)$*, the induced lattice morphism* $\mathbf{S}(R, S) := f_{RS} : (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}), \rho_2) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}), \rho_1)$ *defined by* $f_{RS}(A, B) := (R^{[-1]}A, S^{[-1]}B)$ *is a continuous mapping with respect to the metric topology of the correspondent concept lattices.*

**Proof.** Consider $\varepsilon > 0$ and $(A, B) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$. Then, for every $a \in A$, there is a $\delta > 0$ such that for every $m_2 \in M_2$ with $d_2(a, m_2) < \delta$, we have that $d_1(R^{-1}a, S^{-1}m_2) < \varepsilon$. Take a closed concept $(C, D) \in \underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$ whose distance to $(A, B)$ is less than $\delta$, i.e., $d_2(A, D) < \delta$ and $d_2(C, B) < \delta$. Then, for the chosen $a \in A$, we shall find a $d \in D$ with $d_2(a, d) < \delta$, hence $d_1(R^{-1}a, S^{-1}d) < \varepsilon$. Since $d_1(R^{[-1]}A, S^{[-1]}D) \leq d_1(R^{-1}a, S^{-1}d)$, it follows that $d_1(R^{[-1]}A, S^{[-1]}D) < \varepsilon$. The same holds for $d_1(R^{[-1]}C, S^{[-1]}B)$ concluding that $\rho_1(f_{RS}(A, B), f_{RS}(C, D)) < \varepsilon$, i.e., $f_{RS}$ is continuous. □

**Remark 7.** Analogous arguments shows that if $(R, S)$ is a pseudometric uniformously continuous morphism, then the induced 0-1-lattice homomorphism $f_{RS}$ is uniformously continuous too.

**Proposition 3.10.** *For every continuous pseudometric 0-1-lattice homomorphism* $f : (L_1, \rho_1) \to (L_2, \rho_2)$*, the induced multivalued standard morphism* $(R_f, S_f) : (\mathbb{K}^{\mathcal{T}}(L_2), d_2) \to (\mathbb{K}^{\mathcal{T}}(L_1), d_1)$ *is pseudometric continuous.*

**Proof.** Consider $F_1 \in \mathcal{F}_0(L_1)$ and $I_1 \in \mathcal{I}_0(L_1)$. Then, for every $x \in F_1$, there is a $\delta > 0$ such that for every $y \in L_1$ with $\rho_1(x, y) < \delta$, we have $\rho_2(f(x), f(y)) < \varepsilon$. Then, for every $y \in I_1$ with $\rho_1(x, y) < \delta$, we have $d_1(F_1, I_1) < \delta$. By the definition of $R_f^{-1}F_1$ and $S_f^{-1}I_1$, we obtain that $d_2(R_f^{-1}F_1, S_f^{-1}I_1) < \varepsilon$. □

**Remark 8.** If $f$ is uniformously continuous then $(R_f, S_f)$ is pseudometric uniformously continuous too.

The above results say nothing else than the restriction of the two functors to the pseudometric continuous case are well-defined.

Consider now $\iota : (L, \rho) \to (\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), d)$ defined by $\iota a := (F_a, I_a)$. In order to prove the continuity of $\iota$ we have to show that for every $\varepsilon > 0$ and every $a \in L$, there is a $\delta > 0$ such that for every $b \in L$ with $\rho(a, b) < \delta$, we have $d((F_a, I_a), (F_b, I_b)) < \varepsilon$.

By definition, $d((F_a, I_a), (F_b, I_b)) := \max\{\sigma(F_a, I_b), \sigma(F_b, I_b)\} < \varepsilon$ if and only if $\sigma(F_a, I_b) = \inf \rho(F, I) < \varepsilon$. It follows that there is an $F \in F_a$ and an $I \in I_b$ with $\rho(F, I) < \varepsilon$.

On the other hand, since $a \in F$ and $b \in I$, we conclude that $\rho(F, I) \leq \rho(a, b)$. Choose $\delta := \varepsilon$, hence $\iota$ is continuous. Moreover, since $\delta$ do not depend on $a \in L$, we can conclude that $\iota$ is uniformously continuous. As one can easily see, $\iota$ is not a homeomorphism, hence the categories $\mathbf{T}C_d$ and $\mathbf{L}_d$ are not dual equivalent.

The same holds for $(R_\alpha, S_\beta) : (\mathbb{K}^{\mathcal{T}}, d) \to (\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \rho)$. Since the pseudometric $\rho$ on $\mathbb{K}^{\mathcal{T}}(\underline{\mathcal{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ is the trivial one, and since there are several examples of pseudometrics on $\mathbb{K}^{\mathcal{T}}$ which are not trivial, we conclude that $(R_{\alpha^{-1}}, S_{\beta^{-1}})$, i.e., the inverse of $(R_\alpha, S_\beta)$ in the category $\mathbf{T}opCon_d$ is pseudometric continuous (and even more, pseudometric uniformously continuous), but $(R_\alpha, S_\beta)$ itself is generally not pseudometric continuous.

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