STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLVI**, Number 2, June 2001

CONVOLUTIONS OF PRESTARLIKE FUNCTIONS

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Dedicated to Professor Petru T. Mocanu on his 70^{th} *birthday*

1.Introduction

We denote the class of starlike functions of order α by $S^*(\alpha)$, and the class of convex functions of order α by $K(\alpha)$. The function

$$s_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}} = z + \sum_{n=2}^{\infty} C(\alpha, n) z^n$$

is the well-known extremal function for $S^*(\alpha)$, where

$$C(\alpha, n) = \frac{\prod_{k=2}^{n} (k - 2\alpha)}{(n-1)!} \qquad (n \geq 2).$$

Let $(f * g)(z)$ denote the Hadamard product of two functions $f(z)$ and $g(z)$, that is, if $f(z)$ and $g(z)$ are given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let T denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0), \tag{1}$$

which are analytic in the unit disc $U = \{z \in \mathbf{C} : |z| < 1\}$.

If $f(z)$ is given by (1) and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geq 0),$$

then the Hadamard product of f and g is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $R[\alpha,\beta]$ be a subclass of T, consisting functions which satisfies

$(f * s_\alpha)(z) \in S^*(\beta)$ for $0 \le \alpha < 1$ and $0 \le \beta < 1$. Futher let $C[\alpha,\beta]$ be a subclass of T of functions satisfying $zf'(z) \in R[\alpha,\beta]$ for $0 \le \alpha < 1$ and $0 \le \beta < 1$. $R[\alpha,\beta]$ is called the class of functions α-prestarlike of order β with negative coefficients.

Lemma 1.[7] *Let the function $f(z)$ be defined by* (1). *Then $f(z)$ is in the class $R[\alpha,\beta]$ if and only if*

$$\sum_{n=2}^{\infty}(n-\beta)C(\alpha,n)a_n \le 1-\beta.$$

Lemma 2.[3] *Let the function $f(z)$ be defined by* (1). *Then $f(z)$ is in the class $C[\alpha,\beta]$ if and only if*

$$\sum_{n=2}^{\infty}n(n-\beta)C(\alpha,n)a_n \le 1-\beta.$$

Since $f(z)$ defined by (1) is univalent in the unit disc if $\sum\limits_{n=2}^{\infty} na_n \le 1$; we can see that $f \in R[\alpha,\beta]$ is univalent if $0 \le \alpha \le \frac{1}{2}$; and a function $f(z) \in C[\alpha,\beta]$ is univalent in the unit disc if $0 \le \alpha \le \frac{3-\beta}{2(2-\beta)}$.

Lemma 3.[2, Th.8] *Let $f(z)$ a function defined by* (1) *be in the class $C[\alpha,\beta]$. Then f belongs to the class $R[\alpha,\gamma]$, where*

$$\gamma = \frac{2}{3-\beta}.$$

2.Convolutions

Theorem 1. *If a function $f(z)$ defined by* (1) *belongs to the class $R[\alpha,\beta]$ with $0 \le \beta < 1$ and $0 \le \alpha \le \frac{3-\beta}{2(2-\beta)}$, then $(\underbrace{f * f * ... * f}_{m})(z)$, $m \in \mathbf{N} = \{1,2,...\}$ belongs to the class $R[\alpha,\beta]$, too.*

Proof. Using Lemma 1 we have

$$\sum_{n=2}^{\infty}(n-\beta)C(\alpha,n)a_n^m \le \left[\frac{1-\beta}{2(1-\alpha)(2-\beta)}\right]^{m-1}(1-\beta) \le 1-\beta$$

with $0 \le \beta < 1$ and $0 \le \alpha \le \frac{3-\beta}{2(2-\beta)}$.

Theorem 2. *If a function $f(z)$ defined by* (1) *belongs to the class $C[\alpha,\beta]$ cu $0 \le \beta < 1$ and $0 \le \alpha \le \frac{7-3\beta}{4(2-\beta)}$, then $(\underbrace{f * f * ... * f}_{m})(z) \in C[\alpha,\beta]$,*

$(m \in \mathbf{N})$.

Proof. Using Lemma 2 we have

$$\sum_{n=2}^{\infty} n(n-\beta)C(\alpha,n)a_n^m \leq \left[\frac{1-\beta}{4(1-\alpha)(2-\beta)}\right]^{m-1}(1-\beta) \leq 1-\beta$$

with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{7-3\beta}{4(2-\beta)}$.

THEOREM 3. *Let a function $f(z)$ defined by* (1) *be in the class $R[\alpha,\beta]$ with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$; and let the function $g(z)$ defined by*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_n \geq 0)$$

be in the class $C[\alpha,\beta]$ with $0 \leq \beta < 1$ and $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$. Then we have

$$(\underbrace{f * f * ... * f}_{p} * \underbrace{g * g * ... * g}_{q})(z) \in C[\alpha,\beta], \qquad p,q \in \mathbf{N}.$$

Proof. Applying Lemma 1 and Lemma 2 we have

$$\sum_{n=2}^{\infty} n(n-\beta)C(\alpha,n)a_n^p b_n^q \leq$$

$$\leq \left[\frac{1-\beta}{2(1-\alpha)(2-\beta)}\right]^{p}\left[\frac{1-\beta}{4(1-\alpha)(2-\beta)}\right]^{q-1}(1-\beta) \leq 1-\beta$$

if $0 \leq \beta < 1$, $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$ and $0 \leq \alpha \leq \frac{7-3\beta}{4(2-\beta)}$.

But we have $\frac{3-\beta}{2(2-\beta)} < \frac{7-3\beta}{4(2-\beta)}$, and results $0 \leq \alpha \leq \frac{3-\beta}{2(2-\beta)}$.

We need the following notation

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \qquad (a_{n,i} \geq 0, i = 1,2) \tag{2}$$

and the following results from [1]:

THEOREM 4.[1] *Let the function $f_1(z)$ defined by* (2) *be in the class $R[\alpha,\beta]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta < 1$ and let the function $f_2(z)$ defined by* (2) *be in the class $R[\alpha,\tau]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \tau < 1$. Then $(f_1 * f_2)(z) \in R[\alpha,\psi]$, where*

$$\psi = 1 - \frac{(1-\beta)(1-\tau)}{2(1-\alpha)(2-\beta)(2-\tau) - (1-\beta)(1-\tau)}.$$

The result is sharp for the functions

$$f_1(z) = z - \frac{1-\beta}{2(1-\alpha)(2-\beta)}z^2 \qquad \textit{and} \qquad f_2(z) = z - \frac{1-\tau}{2(1-\alpha)(2-\tau)}z^2.$$

THEOREM 5.[1] *Let the function $f_1(z)$ defined by* (2) *be in the class $C[\alpha,\beta]$ with $0 \le \alpha \le \frac{1}{2}$ and $0 \le \beta < 1$ and let the function $f_2(z)$ defined by* (2) *be in the class $C[\alpha,\tau]$ with $0 \le \alpha \le \frac{1}{2}$ and $0 \le \tau < 1$. Then $(f_1 * f_2)(z) \in C[\alpha,\psi]$, where*

$$\psi = 1 - \frac{(1-\beta)(1-\tau)}{4(1-\alpha)(2-\beta)(2-\tau)-(1-\beta)(1-\tau)}.$$

The result is sharp for the functions

$$f_1(z) = z - \frac{1-\beta}{4(1-\alpha)(2-\beta)}z^2 \qquad \text{and} \qquad f_2(z) = z - \frac{1-\tau}{4(1-\alpha)(2-\tau)}z^2.$$

The following two theorems are generalizations of the Theorem 4 and Theorem 5.

THEOREM 6. *Let the functions $f_i(z)$ $(i = 1, 2, ..., m)$ defined by* (2) *be in the classes $R[\alpha,\beta_i]$ with $0 \le \alpha \le \frac{1}{2}$ and $0 \le \beta_i < 1$ for all $i = 1, 2, ..., m$. Then$(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $R[\alpha,\psi]$, where*

$$\psi = 1 - \frac{\prod_{i=1}^{m}(1-\beta_i)}{2^{m-1}(1-\alpha)^{m-1}\prod_{i=1}^{m}(2-\beta_i) - \prod_{i=1}^{m}(1-\beta_i)}.$$

The result is sharp for the extremal functions defined by

$$f_i(z) = z - \frac{1-\beta_i}{2(1-\alpha)(2-\beta_i)}z^2 \qquad (i = 1, 2, ...m).$$

Proof. We apply the method of the mathematical induction.

For $m = 2$ and $\beta_1 = \beta$, $\beta_2 = \tau$, our theorem is reduced to Theorem 4, which is true. Suppose that

$$f_i(z) \in R[\alpha,\beta_i] \qquad (i = 1, 2, ..., k; k \in \mathbf{N}, k \ge 2) \Rightarrow$$

$$\Rightarrow (f_1 * f_2 * ... * f_k)(z) \in R[\alpha,\psi'],$$

where

$$\psi' = 1 - \frac{\prod_{i=1}^{k}(1-\beta_i)}{2^{k-1}(1-\alpha)^{k-1}\prod_{i=1}^{k}(2-\beta_i) - \prod_{i=1}^{k}(1-\beta_i)}.$$

If $f_{k+1} \in R[\alpha,\beta_{k+1}]$, then from Theorem 4, we have

$$((f_1 * f_2 * ... * f_k) * f_{k+1})(z) \in R[\alpha,\psi],$$

where

$$\psi = 1 - \frac{(1-\psi')(1-\beta_{k+1})}{2(1-\alpha)(2-\psi')(2-\beta_{k+1}) - (1-\psi')(1-\beta_{k+1})},$$

which is equivalent to

$$\psi = 1 - \frac{\prod_{i=1}^{k+1}(1-\beta_i)}{2^k(1-\alpha)^k \prod_{i=1}^{k+1}(2-\beta_i) - \prod_{i=1}^{k+1}(1-\beta_i)}.$$

This means that if the theorem is true for $m = k$, then it is true for $m = k+1$, so that it is true for all $m \geq 2$.

THEOREM 7. *Let the functions $f_i(z)$ $(i = 1, 2, ..., m)$ defined by (2) be in the classes $C[\alpha, \beta_i]$ with $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta_i < 1$ for all $i = 1, 2, ..., m$. Then$(f_1 * f_2 * ... * f_m)(z)$ belongs to the class $C[\alpha, \psi]$, where*

$$\psi = 1 - \frac{\prod_{i=1}^{m}(1-\beta_i)}{4^{m-1}(1-\alpha)^{m-1} \prod_{i=1}^{m}(2-\beta_i) - \prod_{i=1}^{m}(1-\beta_i)}.$$

The result is sharp for the functions

$$f_i(z) = z - \frac{1-\beta_i}{4(1-\alpha)(2-\beta_i)} z^2 \qquad (i = 1, 2, ...m).$$

Proof. For $m = 2$ and $\beta_1 = \beta$, $\beta_2 = \tau$, our theorem is reduced to Theorem 5, which is true.

Suppose that

$$f_i(z) \in C[\alpha, \beta_i] \qquad (i = 1, 2, ..., k; k \in \mathbf{N}, k \geq 2) \Rightarrow$$

$$\Rightarrow (f_1 * f_2 * ... * f_k)(z) \in C[\alpha, \psi'],$$

where

$$\psi' = 1 - \frac{\prod_{i=1}^{k}(1-\beta_i)}{4^{k-1}(1-\alpha)^{k-1} \prod_{i=1}^{k}(2-\beta_i) - \prod_{i=1}^{k}(1-\beta_i)}.$$

If $f_{k+1} \in C[\alpha, \beta_{k+1}]$, then from Theorem 5, we have

$$((f_1 * f_2 * ... * f_k) * f_{k+1})(z) \in C[\alpha, \psi],$$

where

$$\psi = 1 - \frac{(1-\psi')(1-\beta_{k+1})}{4(1-\alpha)(2-\psi')(2-\beta_{k+1}) - (1-\psi')(1-\beta_{k+1})},$$

which is equivalent to

$$\psi = 1 - \frac{\prod_{i=1}^{k+1}(1-\beta_i)}{4^k(1-\alpha)^k \prod_{i=1}^{k+1}(2-\beta_i) - \prod_{i=1}^{k+1}(1-\beta_i)},$$

which means that the theorem is true for all $m \geq 2$.

THEOREM 8. *If* $f(z) \in C[\alpha, \beta_i]$ $(i = 1, 2, ..., m)$ *with*$0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta_i < 1$ *for all* $i = 1, 2, ..., m$, *then*$(f_1 * f_2 * ... * f_m)(z) \in R[\alpha, \tau]$, *where*

$$\tau = 1 - \frac{\prod_{i=1}^{m}(1-\beta_i)}{2 \cdot 4^{m-1}(1-\alpha)^{m-1} \prod_{i=1}^{m}(2-\beta_i) - \prod_{i=1}^{m}(1-\beta_i)}.$$

The result is sharp.

From Theorem 6 (or Theorem 7) and Lemma 3 we obtain the result.

THEOREM 9. *Let the functions* $f_i(z)$ $(i = 1, 2)$ *defined by* (2) *be in the class* $C[\alpha, \beta]$ *with* $0 \leq \alpha \leq \frac{1}{2}$ *and* $0 \leq \beta < 1$. *Then the function* $h(z)$ *defined by*

$$h(z) = z - \sum_{n=2}^{\infty} \left[a_{n,1}^2 + a_{n,2}^2\right] z^n$$

belongs to the class $R[\alpha, \gamma]$, *where*

$$\gamma = 1 - \frac{(1-\beta)^2}{4(1-\alpha)(2-\beta)^2 - (1-\beta)^2}.$$

The result is sharp.

Using Theorem 9 (or Theorem 10) from [1] and Lemma 3 we obtain immediately the result.

References

[1] M. K. Aouf and G. S. Sălăgean, *Certain subclasses of prestarlike functions with negative coefficients*, Studia Univ. Babeş-Bolyai, Math., 39 (1994), no. 1, 19-30.
[2] T. Jakab, *Subclasses of prestarlike functions with negative coefficients* (to appear).
[3] S. Owa and B. A. Uralegaddi, *A class of functions α-prestarlike of order β*, Bull. Korean Math. Soc. 21 (1984), No.2, 77-85.
[4] S. M. Sarangi and B. A. Uralegaddi, *Certain generalization of prestarlike functions with negative coefficients*, Ganita, 34 (1983), 99-105.
[5] T. Sheil-Small, H. Silverman and E. Silvia, *Convolution multipliers and starlike functions*, J. Analyse Math. 41 (1982), 181-192.
[6] H. Silverman and E. Silvia, *Prestarlike functions with negative coefficients*, Internat. J. Math. & Math. Sci. (3) 2 (1979), 427-439.
[7] H. Silverman and E. Silvia, *Subclasses of prestarlike functions*, Math. Japon. 29 (1984), No.6, 929-935.

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