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ON A HYBRID FDTD-MoM TECHNIQUE: 2-D CASE

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Abstract. FDTD techniques offer fast simulations and small memory requirements while MoM is more suitable for free-space field simulations but needs more processing power and memory. A hybrid method that combines the advantages of both would be highly useful in free-space scattering simulation.

1. Introduction

The finite-difference time-domain (FDTD) solution of Maxwell's curl equations is analogous to existing finite-difference solutions of scalar wave propagation and fluid-flow problems in that the numerical model is based upon a direct solution of the governing partial differential equations.

The simplicity and the ability to handle complex geometry make the FDTD method flexible to implement. It is successfully applied for a wide variety of electromagnetic wave interaction problems. FDTD is a nontraditional approach to numerical electromagnetic wave modeling of complex structures for engineering applications, where the method of moments has dominated for many years.

A. Some general characteristic of proposed technique. The goal of this paper is to develop a hybrid technique using FDTD method and MoM technique, combining the benefits of both while ensuring the stability of the method. The analysis is done for a combined two-dimensional conducting and dielectric electromagnetic structure.

We must preserve a certain ratio between the spectral component of the considered impulse with the highest significant frequency and the FDTD grid step, respectively the time step.

This condition must be also respected inside the dielectric.

The FDTD method introduces a non-physical dispersion (artificial, numerical reason) for the phase velocity, dispersion which is more important as the highest frequency component of the incident wave is represented using less points.

For a pure FDTD simulation, this dispersion must behave like a deformation of the original impulse, as small as possible to avoid the reflections on the boundary of the analyzed domain.

The reflection on the boundary of the analyzed domain appear because both the values of incident wave and MoM - computed scattered field suppose an ideal behavior of dispersion while the values resulted from FDTD are affected by the numerical dispersion.

Therefore, a grid not fine enough with respect to the shape and duration of the incident wave impulse leads to unnatural reflections on the boundary, even if inside the domain the FDTD behavior is acceptable.

B. FDTD algorithm - two dimensional case. The field is described by Maxwell's curl equations:

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu}\nabla \times E \tag{1}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon}\nabla \times H - \frac{\sigma}{\varepsilon}E \tag{2}$$

where E is the electric field in volts/meter; H is the magnetic field in amperes/meter; ε is the electrical permittivity in farads/meter; σ is the electrical conductivity in siemens/meter; μ is the magnetic permeability in henrys/meter.

The FDTD algorithm for electromagnetic wave interactions for TM case, with E_z, H_x and H_y field component only:

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu}\frac{\partial E_z}{\partial y} \tag{3}$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu}\frac{\partial E_z}{\partial x} \tag{4}$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right) \tag{5}$$

Then, we use the centered finite-difference expression for the space and time derivatives:

$$F^n(i,j) = F(idx, jdy, ndt)$$

$$\frac{\partial F^n(i,j)}{\partial x} = \frac{F^n(i+1/2,j) - F^n(i-1/2,j)}{dx} + O(dx^2)$$

$$\frac{\partial F^n(i,j)}{\partial t} = \frac{F^{n+1/2}(i,j) - F^{n-1/2}(i-1/2,j)}{dt} + O(dt^2)$$

and finally we have the formula:

$$\psi^{n+4}(i,j) = -dt\frac{\sigma(i,j)}{\varepsilon(i,j)}\psi^{n+3}(i,j) + dt\frac{\sigma(i,j)}{\varepsilon(i,j)}\psi^{n+1}(i,j) - \psi^n(i,j) +$$

$$+\frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dx}\right)^2(\psi^{n+2}(i+1,j) + \psi^{n+2}(i-1,j)) +$$

$$+\frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dy}\right)^2(\psi^{n+2}(i,j+1) + \psi^{n+2}(i,j-1)) +$$

$$+2\left(1 - \frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dx}\right)^2 - \frac{1}{\mu_0\varepsilon(i,j)}\left(\frac{dt}{dy}\right)^2\right)\psi^{n+2}(i,j) \tag{6}$$

where $\psi = E_z$.

To ensure the stability of the time-stepping algorithm, dt must be chosen to satisfy the inequality:

$$c_{\max}dt \leq (1/dx^2 + 1/dy^2)^{-1/2} \tag{7}$$

where $c_{\max}$ is the maximum electromagnetic wave phase velocity within the media being modeled.

2. **Formulation of the problem**

Let us consider a 2D problem with the layout shown in Fig.1. The analyzed domain contains a dielectric zone with arbitrary shape and parameters placed in the proximity of a perfectly conductive material of linear cross-section. The entire system is illuminated with a plane wave propagating towards the origin at an arbitrary angle. The wave will produce secondary scattering waves on the dielectric and will reflect completely on the perfect conductor surface.

We will use during the following computations two boundaries: ∂S_∞ the boundary placed at infinite and ∂S_c united with ∂S_d the boundary of the analyzed

domain. For simplicity of geometry generation, the conductor is parallel with Oy axis, but the method allows any placement of it.

Fig.1. Studied case of scattering problem

3. **Mathematical formulation**

A. General facts. The wave speed in free space: $c = 3 * 10^8 m/s$

The free space permittivity: $\varepsilon_0 = 8.8419 * 10^{-12} F/m$

The free space permeability: $\mu = 4 * \pi * 10^{-7} H/m$

The dielectric conductivity: $\sigma = 10^{-9} S/m$

The dimensions of the analized domain are $L_x = 30m$, $L_y = 20m$.

The incident wave is travelling leftwards in the air. It is one cosine impulse length $iw = 4 * 10^{-9} s$, starting tangential in zero.

The incidence angle of the wave, counterclockwise from Ox axis is $\alpha = \pi/4$.

We must have a fine grid to respect the shape and duration of the incident wave impulse. This condition will prevent the unnatural reflections on the boundary.

The place occupied in space by the impulse (for 45 degrees incidence):

Impulse length: $iw * c = 4 * 10^{-9} * 3 * 10^8 = 1.2s$

The grid step: $dx = dy = dl = 0.1m$

Number of nodes: $iw/(dl * \sqrt{2}) = 1.2/(0.1 * 1.414\ldots) = 8.4$

The phase of the wave, which starts at the far end of domain:

$$Tph = (L_x * \cos(\alpha) + L_y * \sin(\alpha))/c$$

The scattering problem is solved for two shapes of dielectric. More complex structures of dielectric shapes can be solved in a similar mode.

In this paper, the shapes of dielectric analized are cylindrical with rectangular and respectively circular cross-sections. The rectangular dielectric is placed in the centre of analized zone and it has the dimensions $L_x/2m$,respectively $L_y/2m$. The relative permitivity is $\varepsilon_{rel} = 10$. The circular cross-section dielectric is also placed in the centre of the analized zone, with radius $L_y/3m$. The relative permitivity is $\varepsilon_{rel} = 2$.

B. Green's theorem. Derivations of integral equations. For the computation of the field values from the contour we have two Helmholz equations:

$$\nabla^2 \psi_\omega^s + \frac{\omega^2}{c^2} \psi_\omega^s = 0 \tag{8}$$

$$\nabla^2 G_\omega(r, r') + \frac{\omega^2}{c^2} G_\omega(r, r') = -\delta(r - r') \tag{9}$$

where $G_\omega(r, r')$ is the Fourier Transform for Green function, ψ_ω^s is the scattered field and r, r' are the vectors of the position.

$\delta(r - r')$ is Dirac function.

On the conductor, the scattered field is zero: $\psi_\omega^s = 0$.

For the field computation, we apply the Green's theorem in time domain on the area outside $\partial S_d, \partial S_c$ and inside ∂S_∞.

$\psi^s(r,t)$	scattered field
$\psi_0(r,t)$	incident wave
$\psi(r,t)$	total field $\psi(r,t) = E_z(r,t)$
$G_\omega(r,r')$	Fourier Transform for Green function
$G(r,r',t-t')$	2D free-space Green function

$$G(r, r', t - t') = \frac{H(t - t' - u/c)}{\sqrt{\left|(t - t')^2 - \frac{u^2}{c^2}\right|}} \tag{10}$$

$$u = |r - r'|$$

$H(x)$ Heaviside unit step function

$$\iint\limits_{s}(G_{\omega}(r,r')\nabla^2\psi^s_{\omega}(r')-\nabla^2G_{\omega}(r,r')\psi^s_{\omega}(r))dxdy=$$

$$=\int\limits_{\partial S_{\infty}}\left(G_{\omega}(r,r')\frac{\partial\psi^s_{\omega}(r)}{\partial n}-\frac{\partial G_{\omega}(r,r')}{\partial n}\psi^s_{\omega}(r)\right)dl+$$

$$+\int\limits_{\partial S_{d}}\left(G_{\omega}(r,r')\frac{\partial\psi^s_{\omega}(r)}{\partial n}-\frac{\partial G_{\omega}(r,r')}{\partial n}\psi^s_{\omega}(r)\right)dl+$$

$$+\int\limits_{\partial S_{c}}\left(G_{\omega}(r,r')\frac{\partial\psi^s_{\omega}(r)}{\partial n}-\frac{\partial G_{\omega}(r,r')}{\partial n}\psi^s_{\omega}(r)\right)dl \tag{11}$$

The first integral is zero because the field vanishes at infinity.

Considering $\partial S_d+\partial S_c=\partial S$, we have:

$$\iint\limits_{S}(\delta(r-r')\psi^s_{\omega}(r'))dx'dy'=$$

$$\int\limits_{\partial S}\left(G_{\omega}(r,r')\frac{\partial\psi^s_{\omega}(r)}{\partial n}-\frac{\partial G_{\omega}(r,r')}{\partial n}\psi^s_{\omega}(r)\right)dl \tag{12}$$

Now we use the Inverse Fourier Transform to transform the Green function in time domain and we obtain:

$$\psi(r,t)=\psi_0(r,t)+$$

$$+\int\limits_{\partial S_{d}}dl\left(G(r,r',t-t')*\frac{\partial\psi^s(r,t)}{\partial n}-\frac{\partial G(r,r',t-t')}{\partial n}*\psi^s(r,t)\right)+$$

$$+\int\limits_{\partial S_{c}}dl\left(G(r,r',t-t')*\frac{\partial\psi^s(r,t)}{\partial n}\right) \tag{13}$$

The system of integral equations has the number of equations equal with the number of points from discretization made on ∂S_d and ∂S_c. For the points on ∂S_c, the field values $\psi(r,t)$ are zero.

C. Computation of the convolution integrals. For the derivative of G and ψ, we have the expressions:

$$\frac{\partial G(r_c, ndt - t')}{\partial n} = \frac{G_{cont}(i_c, j_c) - G_{int}(i_i, j_i)}{dn};$$
$$\frac{\partial \psi^s(r_c, ndt)}{\partial n} = \frac{\psi^s_{cont}(i_c, j_c) - \psi^s_{int}(i_i, j_i)}{dn}, \tag{14}$$

where G_{cont} is the Green function relative to contour nodes, G_{int} is the Green function relative to interior-contour nodes, ψ_{cont} is the field values on the contour nodes, ψ_{int} is the field values on the interior-contour nodes and the boundary ∂S is equal with $\partial S_d + \partial S_c$.

Considering the symbol $*$ for the convolution operator, the integral becomes:

$$\int_{\partial S} dl \left(G(r, r', t - t') * \frac{\partial \psi^s(r, t)}{\partial n} - \frac{\partial G(r, r', t - t')}{\partial n} * \psi^s(r, t) \right) =$$

$$= \int_{\partial S} dl \frac{1}{dn} (G_{cont}(r, r', t - t') * \psi^s_{int}(r, t) - G_{int}(r, r', t - t') * \psi^s_{cont}(r, t)) \cong$$

$$\cong \sum_{j \in contour} \frac{dl}{dn} (G_{cont}(r, r', t - t') * \psi^s_{int}(r, t)|_j - G_{int}(r, r', t - t') * \psi^s_{cont}(r, t)|_j) \tag{15}$$

The approach from (15) is correct because we know that for an arbitrary function $f(r)$, the contour integral over $f(r)$ means sum over all points on the contour:

$$\int_C f(r) dl \cong \sum_{j \in contour} dl f(r)|_j \tag{16}$$

Moreover, the values for the Green function relative to interior nodes and contour, respectively, do not change at different time steps and they can be computed before the loops begin in the program.

Each node pair may use a finite number of such non-zero Green function values since the Green function for a given $u = |r - r'|$ decays with the inverse of time. Therefore, a precision of 10^{-1} will require approximately ten values of the Green function per node pair.

The definition for the convolution is:

$$G(r, r', t - t') * \psi^s(r, t) = \int_{-\infty}^{t} G(r, r', t - t') \psi^s(r, t') dt' = I(t) \tag{17}$$

We calculate the integral at time $t = (n + 1)dt$, so we will make a notation and then we will write the integral as sum of integrals. The sum can be separated in

three terms. The first two integrals are evaluated and the rest implies the calcul of Green function.

$$I(t)|_{t=(n+1)dt,r} = I^{(n+1)}(r)$$

$$I^{(n+1)}(r) = \int_{ndt+\frac{dt}{2}}^{(n+1)dt} G(r,r',t-t')\psi^s(r,t')dt' +$$

$$+ \int_0^{\frac{dt}{2}t} G(r,r',t-t')\psi^s(r,t')dt' + \sum_{k=1}^{n} \psi^s|_{r,k'} \int_{kdt-\frac{dt}{2}}^{kdt+\frac{dt}{2}t} G(r,r',t-t')dt' \tag{18}$$

The first integral from (18) is zero for all points except for the reference point which we neglect. The second integral from (18) is zero because at the beginning the field value is zero. The last integral is analytically calculated.

For the Green function, we know that:

$$\begin{gathered} G(r,r',t-t') = 0 \text{ for } t-t' < |r-r'|/c; \\ (\text{or } (n+1)dt - t' < |r-r'|/c) \end{gathered} \tag{19}$$

and

$$\begin{gathered} G(r,r',t-t') = 1/\sqrt{(t-t') - |r-r'|^2/c^2}, \\ \text{for } t-t' > |r-r'|/c; \\ (\text{or } (n+1)dt - t' > |r-r'|/c) \end{gathered} \tag{20}$$

4. **Numerical results**

A. Estimation of the error. Estimation of the error arising from neglecting of

$$I_1 = \int_{ndt+dt/2}^{(n+1)dt} G(r,r',t-t')\psi(r,t')dt'$$

form sum (18) has the analytical expression:

$$Er(dt) = \pi c dt/8$$

For example, if the time step $dt = 10^{-10}$ we have

$$Er(dt) = \pi \cdot 3 \cdot 10^8 \cdot 10^{-10}/8 \cong 0.0117$$

The above computing coefficient multiplies the field on the interior contour at time $(n+1)dt$ which is not yet known and will be computed using the values from the whole summation, fed into FDTD method. This leaves us no other choice than to ignore the whole term when computing the scattered field on the contour at time $(n+1)dt$.

As we know, the relation for computing the scattered field suppose the convolution between the wave shape in a point and the correspondent Green function over time. For computational reason (the occupied memory, the time needed for the computing) the sum which calculates this integral must be truncated. The sum truncation is another error source that can lead to parasite reflections on the boundary or to a reduction of the response induced by the conductor on the system.

B. Analysis of the Stability. For the test of the stability, we must run the program for different steps in space and time. Because the results must be comparable, we must take the measure to fix the initial conditions of the simulation.

The elements that are fixed (must be fixed) are:

1. the dimension of the analyzed domain
2. the absolute position of the conductor
3. the dimension of the conductor
4. the absolute position of the dielectric
5. the absolute dimension of the dielectric
6. the position of the wave front at starting moment of the simulation.

The absolute position of the conductor doesn't pose any problem being fixed. Also the position of the wave front could be chosen at point (L_x, L_y) at moment $t = 0$.

The simulation implementations in the program groups the sum terms in order to take advantage of the Green function values duplications for pairs of points placed at same relative distance.

Let us consider for example the tested geometry. The domain has 31×21 points (grid values) and the conductor has 21×2 points (grid values) which normally would require to compute the Green function for $2 \times 140 \times 140$ points rising to approximately 39.000 sets of values.

Using the fact that the points on the contour are regularly spaced and the distances between the pairs of points are repeated, the number of Green function to compute and, of course, to store is reduced to approximately only 770.

This significant reduction of the necessary memory (of about 50 times), allows simulations with a step seven times finer ($7 \times 7 \sim = 50$) than without the optimization.

The price paid for these optimizations is solving the problem on a grid with the step on x equal with the step on y, and uniform distributed on the boundary.

The advantages obtained by indexed computations of Green function remain valid, but in a lesser way, for non-equal x and y steps, uniform distributed.

The stability of the method is proven by Fig.2 and Fig.3, containing field plots over space at the same moment of time but being computed with different space steps. The same stands true for Fig.4 and Fig.5 in the circular section case.

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