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PERTURBATIONS OF CERTAIN NONLINEAR PARTIAL FINITE DIFFERENCE EQUATIONS

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Abstract. In the present paper we establish some new variation of constants formulae for nonlinear perturbed partial finite difference equations in two independent variables. We also present some applications to convey the importance of our results in the qualitative theory of certain partial finite difference equations.

1. Introduction

During the past few years the abundance of applications is stimulating a rapid development of the theory of finite difference equations. A variety of new methods and tools are developed by different investigators to study the various types of finite difference equations. In the theory of ordinary finite difference equations the method of variation of parameters is a very useful tool in studying the properties of solutions of perturbed finite difference equations. Motivated and inspired by the results given in [5], see also [1-4, 6-10], in the present paper we establish some representation formulae related to the solutions of a certain nonlinear partial finite difference equation and its perturbed partial finite difference equation in two independent variables. We also use these formulae to study certain properties of the solutions of the corresponding perturbed partial finite difference equation.

2. Statement of results

In what follows, we let $N_0 = \{0, 1, 2, \dots\}$, and

$$N(x_0) = \{x_0, x_0 + 1, x_0 + 2, \dots\}, \quad N(y_0) = \{y_0, y_0 + 1, y_0 + 2, \dots\},$$

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for x_0, y_0 in N_0. The empty sums and products are taken to be 0 and 1 respectively. For any functions $z(x,y), w(m,n), z(x,y,w(m,n))$, x,y,m,n in N_0, we define

$$\Delta_1 z(x,y) = z(x+1,y) - z(x,y), \quad \Delta_2 z(x,y) = z(x,y+1) - z(x,y),$$

$$\Delta_2\Delta_1 z(x,y) = \Delta_1 z(x,y+1) - \Delta_1 z(x,y)),$$

$$\Delta_m z(x,y,w(m,n)) = z(x,y,w(m+1,n)) - z(x,y,w(m,n)),$$

$$\Delta_n z(x,y,w(m,n)) = z(x,y,w(m,n+1)) - z(x,y,w(m,n)).$$

We denote the product $N(x_0)\times N(y_0)$ by $N(x_0,y_0)$. For $(x_0,y_0),(x,y)$ in $N(x_0,y_0)$ we define

$$\phi(x,y,x_0,y_0,w(x,y)) = \Delta_w z(x,y,x_0,y_0,w(x,y)),$$

where

$$\Delta_w z(x,y,x_0,y_0,w(x,y))(w(x+1,y) - w(x,y))) =$$

$$= z(x,y,x_), y_0, w(x+1,y)) - z(x,y,x_0,y_0,w(x,y)).$$

We consider the nonlinear partial finite difference equation

$$\Delta_1\Delta_1 u(x,y) = f(x,y,u(x,y)), \quad u(x,y_0) = u(x_0,y) = u_0, \tag{E}$$

and its perturbed nonlinear finite difference equation

$$\Delta_2\Delta_1 v(x,y) = f(x,y,v(x,y)) + g(x,y,v(x,y)), \quad v(x,y_0) = v(x_0,y) = u_0, \tag{P}$$

for (x,y) in $N(x_0,y_0)$, where u,v are real-valued functions defined on $N(x_0,y_0)$, f,g are real-valued functions defined on $N(x_0,y_0)\times R$, R denotes the set of real numbers, and u_0 is a constant. We use $u(x,y,x_0,y_0,u_0)$ and $v(x,y,x_0,y_0,u_0)$ to denote the solutions of (E) and (P) respectively passing through the point $x(x_0,y_0)\in N(x_0,y_0)$.

A useful nonlinear variation of constants formula is established in the following theorem.

Theorem 1. *Suppose that $u(x,y,x_0,y_0,u_0)$ is the unique solution of (E) and $\phi(x+1,y,x_0,y_0,w(x,y))$, $\phi^{-1}(x+1,y,x_0,y_0,w(x,y))$ exist for (x,y) in $N(x_0,y_0)$. Then any solution $v(x,y,x_0,y_0,u_0)$ of (P) satisfies the relation*

$$v(x,y,x_0,y_0,u_0) = u(x,y,x_0,y_0,u_0 + \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1}\phi^{-1}(s+1,t,x_0,y_0,w(s,t))\times$$
$$\times[A(s,t,x_0,y_0,w(s,t)) + g(s,t,v(s,t,x_0,y_0,u_0))]) \tag{2.1}$$

where $w(x,y)$ is a solution of the equation

$$\Delta_2\Delta_1 w(x,y) = \phi^{-1}(x+1,y,x_0,y_0,w(x,y))[A(x,y,x_0,y_0,w(x,y))+$$

$$+g(x,y,v(x,y,x_0,y_0,u_0))], \quad w(x,y_0) = w(x_0,y) = u_0, \tag{2.2}$$

for (x,y) in $N(x_0,y_0)$ and

$$A(x,y,x_0,y_0,w(x,y)) = -\{[\Delta_1 u(x,y+1,x_0,y_0,w(x,y+1))-$$

$$-\Delta_1 u(x,y+1,x_0,y_0,w(x,y))] + [\Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1))-$$

$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))]\Delta_1 w(x,y+1)\}, \tag{2.3}$$

for $(x,y) \in N(x_0,y_0)$.

Another interesting and useful representation formula is given in the following theorem.

Theorem 2. *Suppose that $u(x,y,x_0,y_0,u_0)$ is the unique solution of (E) and $\phi(x+1,y,x_0,y_0,w(x,y))$, $\phi^{-1}(x+1,y,x_0,y_0,w(x,y))$ exist for $(x,y) \in N(x_0,y_0)$. Then any solution $v(x,y,x_0,y_0,u_0)$ of (P) satisfies the relation*

$$v(x,y,x_0,y_0,u_0) = u(x,y,x_0,y_0,u_0) + \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} B(x,y,x_0,y_0,w(s,t))+$$

$$+\sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} \phi(x,y,x_0,y_0,w(x,y))\times$$

$$\times\phi^{-1}(s+1,t,x_0,y_0,w(s,t))[A(s,t,x_0,y_0,w(s,t))+$$

$$+g(s,t,v(s,t,x_0,y_0,u_0))], \tag{2.4}$$

where $A(x,y,x_0,y_0,w(x,y))$ is given by (2.3) and

$$B(x,y,x_0,y_0,w(s,t)) = [\Delta_w u(x,y,x_0,y_0,w(s,t+1))-$$

$$-\Delta_w u(x,y,x_0,y_0,w(s,t))]\Delta_1 w(s,t+1), \tag{2.5}$$

where $w(x,y)$ is a solution of (2.2).

3. **Proof of Theorem 1**

Since $u(x,y,x_0,y_0,u_0)$ is the solution of (E), by the method of variation of parameters we can find the solution of (P) by the relation

$$v(x,y,x_0,y_0,u_0) = u(x,y,x_0,y_0,w(x,y)), \quad w(x_0,y) = w(x,y_0) = u_0, \tag{3.1}$$

where the function $w(x,y)$ is yet to be determined. For this it is necessary that

$$\Delta_1 v(x,y,x_0,y_0,u_0) = u(x+1,y,x_0,y_0,w(x+1,y)) - u(x,y,x_0,y_0,w(x,y)) =$$
$$= \Delta_1 u(x,y,x_0,y_0,w(x,y)) +$$
$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y). \tag{3.2}$$

From (3.2) we have

$$\Delta_2\Delta_1 v(x,y,x_0,y_0,u_0) = \Delta_1 u(x,y+1,x_0,y_0,w(x,y+1)) -$$
$$-\Delta_1 u(x,y,x_0,y_0,w(x,y)) + \Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1))\Delta_1 w(x,y+1) -$$
$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y) =$$
$$= \Delta_1 u(x,y+1,x_0,y_0,w(x,y)) - \Delta_1 u(x,y,x_0,y_0,w(x,y)) +$$
$$+\Delta_1 u(x,y+1,x_0,y_0,w(x,y+1)) - \Delta_1 u(x,y+1,x_0,y_0,w(x,y)) +$$
$$+\Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1))\Delta_1 w(x,y+1) -$$
$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y+1) +$$
$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y+1) -$$
$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_1 w(x,y) =$$
$$= \Delta_2\Delta_1 u(x,y,x_0,y_0,w(x,y)) + \{[\Delta_1 u(x,y+1,x_0,y_0,w(x,y+1)) -$$
$$-\Delta_1 u(x,y+1,x_0,y_0,w(x,y))] + [\Delta_w u(x+1,y+1,x_0,y_0,w(x,y+1)) -$$
$$-\Delta_w u(x+1,y,x_0,y_0,w(x,y))]\Delta_1 w(x,y+1)\} +$$
$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_2\Delta_1 w(x,y) =$$
$$= \Delta_2\Delta_1 u(x,y,x_0,y_0,w(x,y)) - A(x,y,x_0,y_0,w(x,y)) +$$
$$+\Delta_w u(x+1,y,x_0,y_0,w(x,y))\Delta_2\Delta_1 w(x,y). \tag{3.3}$$

Now from (E), (P) and (3.3) we have

$$f(x,y,v(x,y,x_0,y_0,u_0)) + g(x,y,v(x,y,x_0,y_0,u_0)) =$$

$$= f(x, y, u(x, y, x_0, y_0, w(x, y)) - A(x, y, x_0, y_0, w(x, y))+$$

$$+\phi(x + 1, x_0, y_0, w(x, y))\Delta_2\Delta_1 w(x, y),$$

which because of (3.1) and the fact that $\phi^{-1}(x + 1, y, x_0, y_0, w(x, y))$ exists, reduces to

$$\Delta_2\Delta_1 w(x, y) = \phi^{-1}(x + 1, y, x_0, y_0, w(x, y))[A(x, y, x_0, y_0, w(x, y))+$$

$$+g(x, y, v(x, y, x_0, y_0, u_0))], \quad w(x_0, y) = w(x, y_0) = u_0, \tag{3.4}$$

which determined the required function $w(x, y)$. The solutions of (3.4) then determine $w(x, y)$. Further from (3.4) we have

$$w(x, y) = u_0 + \sum_{s=x_0}^{x-1} \sum_{t=y_0}^{y-1} \phi^{-1}(s + 1, t, x_0, y_0, w(s, t))[A(s, t, x_0, y_0, w(s, t))+$$

$$+g(s, t, v(s, t, x_0, y_0, w(s, t))]. \tag{3.5}$$

From (3.5) and (3.1), (2.1) is immediate. The proof is complete.

4. **Proof of Theorem 2**

For $x_0 \le m \le x$, $y_0 \le n \le y$, $x_0, m, x \in N(x_0)$, $y_0, n, y \in N(y_0)$, we have

$$\Delta_m u(x, y, x_0, y_0, w(m, n)) = u(x, y, x_0, y_0, w(m + 1, n)) - u(x, y, x_0, y_0, w(m, n)) =$$

$$= \Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n). \tag{4.1}$$

From (4.1) we have

$$\Delta_n\Delta_m u(x, y, x_0, y_0, w(m, n)) = \Delta_w u(x, y, x_0, y_0, w(m, n + 1))\Delta_1 w(m, n + 1)-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n) =$$

$$= \Delta_w u(x, y, x_0, y_0, w(m, n + 1))\Delta_1 w(m, n + 1)-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n + 1)+$$

$$+\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n + 1)-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n) =$$

$$= [\Delta_w u(x, y, x_0, y_0, w(m, n + 1))-$$

$$-\Delta_w u(x, y, x_0, y_0, w(m, n))]\Delta_1 w(m, n + 1)+$$

$$+\Delta_w u(x, y, x_0, y_0, w(m, n))\Delta_2\Delta_1 w(m, n) =$$

$$= B(x, y, x_0, y_0, w(m, n))\Delta_1 w(m, n + 1)+$$

$$+\phi(x,y,x_0,y_0,w(m,n))\Delta_2\Delta_1 w(m,n). \tag{4.2}$$

Now keeping x, y, m fixed in (4.2), set $n = t$ and sum over t from y_0 to $y-1$, and then keeping x, y, t fixed in the resulting inequality, set $m = s$ and sum over s from x_0 to $x-1$, to obtain

$$u(x,y,x_0,y_0,w(x,y)) = u(x,y,x_0,y_0,u_0) + \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} B(x,y,x_0,y_0,w(s,t))+$$

$$+\sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} \phi(x,y,x_0,y_0,w(s,t))\Delta_2\Delta_1 w(s,t). \tag{4.3}$$

If $w(x,y)$ is any solution of (2.2), then the result (2.4) follows from (4.3), (3.1) and (2.2). The proof is complete.

5. **Some applications**

In this section we use the formulae given in Theorems 1 and 2 to study the boundedness of the solutions of perturbed finite difference equation (P) under some suitable conditions on the functions involved in (P). We say that the solution $u(x,y,x_0,y_0,u_0)$ of (E) is globally uniformly stable if there exists a constant $M > 0$ such that $|u(x,y,x_0,y_0,u_0)| \le M|u_0|$, for $f(x,y) \in N(x_0,y_0)$ and $|u_0| < \infty$.

We shall need the following special version of the inequality established be Pachpatte in [8,Theorem 1].

Lemma. *Let $u(x,y)$ and $h(x,y)$ be real-valued nonnegative functions defined on N_0^2 and $c \ge 0$ be a constant. If*

$$u(x,y) \le c + \sum_{s=0}^{x-k}\sum_{t=0}^{y-1} h(s,t)u(s,t),$$

for $x, y \in N_0$, then

$$u(x,y) \le c\prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1} h(s,t)\right],$$

for $x, y \in N_0$.

We first give the following application of the variation of constants formula established in Theorem 1.

Theorem 3. *Let the solution $u(x, y, x_0, y_0, u_0)$ of (E) be globally uniformly stable and the hypothesis of Theorem 1 hold. Further, suppose that*

$$|\phi^{-1}(x+1, y, x_0, y_0, w(x, y))[A(x, y, x_0, y_0, w(x, y))+$$

$$+g(x, y, v(x, y, x_0, y_0, u_0))| \leq p(x, y)|w(x, y)|], \tag{5.1}$$

for $(x, y) \in N(x_0, y_0)$ where $p(x, y)$ is a real-valued nonnegative function defined on $N(x_0, y_0)$ and

$$\prod_{s=x_0}^{x-1} \left[1 + \sum_{t=y_0}^{y-1} p(s, t)\right] < \infty, \tag{5.2}$$

for $(x, y) \in N(x_0, y_0)$. Then any solution $v(x, y, x_0, y_0, u_0)$ to (P) is bounded for $(x, y) \in N(x_0, y_0)$.

Proof. By Theorem 1, any solution $v(x, y, x_0, y_0, u_0)$ of (P) satisfies

$$v(x, y, x_0, y_0, u_0) = u(x, y, x_0, y_0, w(x, y)), \quad w(x, y_0) = w(x_0, y) = u_0, \tag{5.3}$$

where $w(x, y)$ is given by (3.5) is a solution of (2.2). Using (3.5) and (5.1) we have

$$|w(x, y)| \leq |u_0| + \sum_{s=x_0}^{x-1} \sum_{t=y_0}^{y-1} |\phi^{-1}(s+1, t, x_0, y_0, w(x, t)) \times$$

$$\times [A(s, t, x_0, y_0, w(s, t)) + g(s, t, v(s, t, x_0, y_0, y_0))]| \leq$$

$$\leq |u_0| + \sum_{s=x_0}^{x-1} \sum_{t=y_0}^{y-1} p(s, t)|w(s, t)|. \tag{5.4}$$

Now a suitable application of Lemma to (5.4) yields

$$|w(x, y)| \leq |u_0| \prod_{s=x_0}^{x-1} \left[1 + \sum_{t=y_0}^{y-1} p(s, t)\right]. \tag{5.5}$$

The right hand side of (5.5) can be made sufficiently small by using (5.2) and assuming that $|u_0|$ is sufficiently small, i.e.

$$|w(x, y)| \leq \varepsilon, \tag{5.6}$$

where $\varepsilon > 0$ is arbitrary, constant. From (5.3) we have

$$|v(x, y, x_0, y_0, u_0)| = |u(x, y, x_0, y_0, w(x, y)| \tag{5.7}$$

From the global uniform stability of the solution $u(x, y, x_0, y_0, u_0)$ of (E) and (5.6) and (5.7) we have

$$|v(x, y, x_0, y_0, u_0)| \leq M\varepsilon,$$

which implies the boundedness of the solution of (P). The proof is complete.

We next give the following application of the variation of constants formula established in Theorem 2.

Theorem 4. *Assume that the hypotheses of Theorem 2 hold and the functions involved in (2.4) satisfy*

$$\sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1}|B(x,y,x_0,y_0,w(x,y))| \le M_1, \tag{5.8}$$

$$|\phi(x,y,x_0,y_0,w(x,y))\phi^{-1}(s+1,t,x_0,y_0,w(s,t))| \le M_2, \tag{5.9}$$

$$|A(x,y,x_0,y_0,w(x,y)) + g(x,y,v(x,y,x_0,y_0,u_0))| \le$$

$$\le p(x,y)|v(x,y,x_0,y_0,u_0)|, \tag{5.10}$$

where M_1 and M_2 are nonnegative constants and $p(x,y)$ is a real-valued nonnegative function defined on $N(x_0,y_0)$ and

$$\prod_{s=x_0}^{x-1}\left[1+\sum_{t=y_0}^{y-1}p(s,t)\right] < \infty, \tag{5.11}$$

for $(x,y)\in N(x_0,y_0)$. Then for every bounded solution $u(x,y,x_0,y_0,u_0)$ of (E) for $(x,y)\in N(x_0,y_0)$, the corresponding solution $v(x,y,x_0,y_0,u_0)$ of (P) is bounded for $(x,y)\in N$.

The proof of this theorem follows by using (5.8)-(5.10) in (2.4) and applying Lemma and condition (5.11). Here we omit the details.

We note that the results given in Theorem 1-4 can very easily be extended when the perturbation term g involved in (P) is of the more general type i.e. when the equation (P) is of the form

$$\Delta_2\Delta_1 v(x,y) = f(x,y,v(x,y)) + g(x,y,v(x,y),Tv(x,y)),$$

$$v(x,y_0) = v(x_0,y) = u_0, \tag{P'}$$

where

$$Tv(x,y) = \sum_{s=x_0}^{x-1}\sum_{t=y_0}^{y-1} h(x,y,s,t,v(s,t)).$$

The formulations of such results corresponding to the equations (E) and (P') are very close to that of the results given in the above theorems with suitable modifications and hence we do not discuss the details.

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