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A KOROVKIN-TYPE THEOREM FOR THE APPROXIMATION OF n-VARIATE B-CONTINUOUS FUNCTIONS

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Abstract. The aim of this note is to extend the results from [1], [4] to the case of n variate B-continuous functions in the sense of Bogel [5]. In the section 1 we present the notions of n-variate B-continuous function and uniform n-variate B-continuous function. Some relationship among these notions are also presented. In the section 2, we discuss a Korovkin-type criterion for the approximation by means of linear positive operators of the B-continuous functions of n-variables. The main result of the paper is the theorem 2.1. In the section 3 we present some applications of the theorem 2.1.

1. Let $\mathbf{R}^{I^n}$ be the space of functions $f : I^n \to \mathbf{R}$, where I=[0,1] and n is a positive integer. The notion of B-continuous function was introduced in [5] using the operator $\Delta_2 : \mathbf{R}^{I^2} \to \mathbf{R}^{I^2}$

$$\Delta_2 [f; M, M'] = \Delta_{s_1,s_2} [f; M, M'] =$$

$$= f(s_1, s_2) - f(s_1, x_2) - f(x_1, s_2) + f(x_1, x_2) \tag{1.1}$$

for any $f \in \mathbf{R}^{I^2}$ and any points $M(x_1, x_2), M'(s_1, s_2) \in I^2$.

Let $\Delta_2 : \mathbf{R}^I \to \mathbf{R}^I$ be the univariate operator given by

$$\Delta_2 [f; M, M'] = \Delta_{s_1} [f; x_1] = f(s_1) - f(x_1) \tag{1.2}$$

for any $f \in \mathbf{R}^I$ and any points $M(x_1), M'(s_1) \in I$.

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If $f \in \mathbf{R}^{I^2}$ and ${}_{s_1}\Delta, {}_{s_2}\Delta$ are the parametric extension of the operator (1,2), then the following equality holds:

$$\Delta_{s_1,s_2}[f;x_1,x_2] = ({}_{s_1}\Delta \circ {}_{s_2}\Delta)[f;x_1,x_2]. \tag{1.3}$$

The last remark allows us to define the operator of n-variate difference by

Definition 1.1: Let $f \in \mathbf{R}^{I^n}$ be a given function and ${}_{s_1}\Delta, \ldots, {}_{s_n}\Delta$ be the parametric extensions of the operator (1,2). The operator $\Delta_n : \mathbf{R}^{I^n} \to \mathbf{R}^{I^2}$ given by

$$\Delta_n[f;M,M'] = \Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n] = ({}_{s_1}\Delta \circ \ldots \circ_{s_n}\Delta)[f;x_1,\ldots,x_n] \tag{1.4}$$

for any functions $f \in \mathbf{R}^{I^n}$ and any points $M(x_1,\ldots,x_n), M'(s_1,\ldots s_n) \in I^n$ is called operator of n-variate difference.

Remark 1.1: It is easy to see that the representation

$$\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_2] = f(s_1,\ldots,s_n) - \sum_{i=1}^{n} f(x_1,\ldots,x_{i-1},s_i,x_{i+1},\ldots,x_n) +$$

$$+ \sum_{i,j=1}^{n} f(x_1,\ldots,x_{i-1},s_i,x_{i+1},\ldots,x_{j-1},s_j,x_{j+1},\ldots,x_n) - \ldots + (-1)^n f(x_1,\ldots,x_n) \tag{1.5}$$

is valid.

Definition 1.2: The function $f \in \mathbf{R}^{I^n}$ is called B-continuous in the point $M(x_1,\ldots,x_n) \in I^n$ is the equality

$$\lim_{(s_1,\ldots s_n) \to (x_1,\ldots,x_n)} \Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_2] = 0 \tag{1.6}$$

holds.

If $f \in \mathbf{R}^{I^n}$ is B-continuous at every point of I^n one says that f is B-continuous on I^n and the set of all B-continuous functions on I^n is denoted by $C_b(I^n)$.

Definition 1.3: The function $f \in \mathbf{R}^{I^n}$ is uniform B-continuous on I^n if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ so that for any point $(x_1,\ldots,x_n), (s_1,\ldots s_n) \in I^n$ for which one has

$$|x_1 - s_1| < \delta, \ldots, |x_n - s_n| < \delta \tag{1.7}$$

the inequality

$$\left|\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_2\right]\right| < \varepsilon \tag{1.8}$$

holds.

Definition 1.4: The function $f \in \mathbf{R}^{I^n}$ is B-bounded on I^n if there exists a positive number K so that:

$$\left|\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_2\right]\right| \leq K, (\forall)\left(x_1,\ldots,x_n\right), \left(s_1,\ldots s_n\right) \in I^n \tag{1.9}$$

The relationships between B-continuous, uniform B-continuous and B-bounded functions are immediately and are contained in the following two lemmas:

Lemma 1.1: If $f \in C_b\left(I^n\right)$ then f is uniform B-continuous on I^n.

Lemma 1.2: If $f \in C_b\left(I^n\right)$ then f is B-bounded on I^n.

2. We will establish a Korovkin type theorem for the approximation on $C_b\left(I^n\right)$. First, we establish an auxiliary result.

Lemma 2.1: Let $f \in C_b\left(I^n\right)$ be arbitrarily chosen. For any positive number $\varepsilon > 0$ there exist n positives numbers $A_i = A_i(\varepsilon), i = \overline{1,n}$ so that for any $\left(x_1,\ldots,x_n\right), \left(s_1,\ldots s_n\right) \in I^n$ one has:

$$\left|\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_n\right]\right| \leq \frac{\varepsilon}{n+1} + \sum_{i=1}^{n} A_i(\varepsilon)\left(x_i - s_i\right)^2. \tag{2.1}$$

Proof. Because $f \in C_b\left(I^n\right)$, from lemma 1.1 it follows that f is uniform B-continuous on I^n i.e. for $\left(x_1,\ldots,x_n\right), \left(s_1,\ldots s_n\right) \in I^n$ with $\left|x_1 - s_1\right| < \delta(\varepsilon), \ldots, \left|x_n - s_n\right| < \delta(\varepsilon)$ one has

$$\left|\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_n\right]\right| < \frac{\varepsilon}{n+1}. \tag{2.2}$$

Let $\varepsilon > 0$ be a given positive number and $\left(x_1,\ldots,x_n\right), \left(s_1,\ldots s_n\right) \in I^n$. The inequalities $\left|x_i - s_i\right| < \delta(\varepsilon)$ can be valid for all $i \in \{1,2,\ldots,n\}$, for $(n-1)$ values of $i \in \{1,2,\ldots,n\}$, ..., for one value of $i \in \{1,2,\ldots,n\}$ or for none of the values $i \in \{1,2,\ldots,n\}$.

If $\left|x_i - s_i\right| < \delta(\varepsilon)$ for any $i \in \{1,2,\ldots,n\}$, from (2.2) one deduces that

$$\left|\Delta_{s_1,\ldots,s_n}\left[f;x_1,\ldots,x_n\right]\right| < \frac{\varepsilon}{n+1} \tag{2.3}$$

Because $f \in C_b(I^n)$, from lemma 1.2 it follows that there exists a positive number K such that

$$|\Delta_{s_1,\dots,s_n}[f;x_1,\dots,x_n]| \le K. \tag{2.4}$$

We suppose that there is only one value $j \in \{1,2,\dots,n\}$ so that $|x_j - s_j| \ge \delta(\varepsilon)$.

If j = 1 then $|x_1 - s_1| \ge \delta(\varepsilon), |x_2 - s_2| < \delta(\varepsilon), \dots, |x_n - s_n| < \delta(\varepsilon)$.

For $(x_1,\dots,x_n),(s_1,\dots s_n) \in I^n$ with these properties we have

$$|\Delta_{s_1,\dots,s_n}[f;x_1,\dots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2}(x_1 - s_1)^2. \tag{2.5}$$

This way, for the points $(x_1,\dots,x_n),(s_1,\dots s_n) \in I^n$ for which there is only one value $j \in \{1,2,\dots,n\}$ such that $|x_j - s_j| \ge \delta(\varepsilon)$ we have:

$$|\Delta_{s_1,\dots,s_n}[f;x_1,\dots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2}\sum_{i=1}^{n}(x_i - s_i)^2. \tag{2.6}$$

In a similar way, for the points $(x_1,\dots,x_n),(s_1,\dots s_n) \in I^n$ for which there are only two values $i,j \in \{1,2,\dots,n\}$, $i \ne j$ such that $|x_i - s_i| \ge \delta(\varepsilon), |x_j - s_j| \ge \delta(\varepsilon)$, we have:

$$|\Delta_{s_1,\dots,s_n}[f;x_1,\dots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2^2}\sum_{i,j=1,i\ne j}^{n}(x_i - s_i)^2(x_j - s_j)^2 \tag{2.7}$$

For all the points $(x_1,\dots,x_n),(s_1,\dots s_n) \in I^n$ for which $|x_i - s_i| \ge \delta(\varepsilon)$, $(\forall)\, j \in \{1,2,\dots,n\}$ we have:

$$|\Delta_{s_1,\dots,s_n}[f;x_1,\dots,x_n]| \le K \cdot [\delta(\varepsilon)]^{-2^n}(x_1 - s_1)^2 \dots (x_n - s_n)^2. \tag{2.8}$$

With these observations, for any $(x_1,\dots,x_n),(s_1,\dots s_n) \in I^n$ the next relation holds:

$$\begin{aligned}|\Delta_{s_1,\dots,s_n}[f;x_1,\dots,x_n]| \le{}& \frac{\varepsilon}{n+1} + K \cdot [\delta(\varepsilon)]^{-2}\sum_{i=1}^{n}(x_i - s_i)^2 + \\ &+K \cdot [\delta(\varepsilon)]^{-2^2}\sum_{i,j=1,i\ne j}^{n}(x_i - s_i)^2(x_j - s_j)^2 + \dots + \\ &+K \cdot [\delta(\varepsilon)]^{-2^n}(x_1 - s_1)^2 \dots (x_n - s_n)^2\end{aligned} \tag{2.9}$$

Because $s_k, x_k \in [0,1]$ $(\forall)\, k \in \{1,2,\ldots,n\}$ we have that $(x_k - s_k)^2 \leq 1$.

Using this observation and (2.9), one obtains:

$$|\Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_n]| \leq$$

$$\leq \frac{\varepsilon}{n+1} + K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2}\right\}(x_1-s_1)^2+$$

$$+K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2^2}\right\}(x_2-s_2)^2+\ldots+$$

$$+K\cdot[\delta(\varepsilon)]^{-2}(x_n-s_n)^2. \tag{2.10}$$

Choosing then

$$A_{1=}K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2}\right\}$$
$$A_{2=}K\cdot[\delta(\varepsilon)]^{-2}\left\{1+[\delta(\varepsilon)]^{-2}+\ldots+[\delta(\varepsilon)]^{-2^n+2^2}\right\}$$
$$\ldots$$
$$A_{n=}K\cdot[\delta(\varepsilon)]^{-2}$$

it follows that (2.1) is valid.

Now we can establish the main result of the paper. We consider the following functions on I^n :

$$e_0(s_1,\ldots s_n)=1, e_i(s_1,\ldots s_n)=s_i, i=\overline{1,n}\ ,(s_1,\ldots s_n)\in I^n.$$

Theorem 2.1: *Let* $\{L_{m_1,m_2,\ldots,m_n}\}$ *be a sequence of positive linear operators mapping the functions of* R^{I^n} *into functions of* R^{I^n} *such that for all* $(x_1,\ldots x_n)\in I^n$ *one has*

i) $L_{m_1,m_2,\ldots,m_n}(e_0;x_1,\ldots x_n)=1$;

ii) $L_{m_1,m_2,\ldots,m_n}(e_0;x_1,\ldots x_n)=x_i+\alpha^{(i)}_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)$,

$i\in\{1,2,\ldots,n\}$;

iii) $L_{m_1,m_2,\ldots,m_n}\left(\sum\limits_{i=1}^{n} e_i^2;x_1,\ldots x_n\right)=\sum\limits_{i=1}^{n} x_i^2+\delta_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)$,

where the sequences $\left\{\alpha^{(i)}_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)\right\}$, $\{\delta_{m_1,m_2,\ldots,m_n}(x_1,\ldots x_n)\}$ tend to zero uniformly on I^n as $m_1,m_2,\ldots m_n$ tend to infinity.

If $f(\cdot,\ldots,\cdot)\in C_b(I^n)$, we introduce the notation:

$(\star)$ $U_{m_1,m_2,\ldots,m_n}(f;x_1,\ldots x_n)=L_{m_1,m_2,\ldots,m_n}(f(x_1,\ldots x_n)-\Delta_{\cdot,\ldots,\cdot}[f;x_1,\ldots x_n])$

In the hypothesis i), ii), iii) the sequence $\{U_{m_1,m_2,\ldots,m_n}(f)\}$ converges to f uniformly on I^n, for any $f \in C_b(I^n)$.

Proof. It is obvious that $U_{m_1,m_2,\ldots,m_n}$ is a well-defined operator on $C_b(I^n)$. Let $f \in C_b(I^n)$ be arbitrarily chosen, $(x_1,\ldots x_n) \in I^n$ and $\varepsilon > 0$ given.

Because $L_{m_1,m_2,\ldots,m_n}$ is a linear operator reproducing the constant functions (from the condition i)), we have:

$$f\ (x_1,\ldots x_n) - U_{m_1,m_2,\ldots,m_n}(f;x_1,\ldots x_n) =$$
$$= L_{m_1,m_2,\ldots,m_n}(\ \Delta_{\cdot,\ldots,\cdot}[f;x_1,\ldots x_n]) \tag{2.11}$$

From the positivity of $L_{m_1,m_2,\ldots,m_n}$ we have

$$|L_{m_1,m_2,\ldots,m_n}(\ g\,;x_1,\ldots x_n)| =$$

$$\max\{L_{m_1,m_2,\ldots,m_n}(\ g;\ x_1,\ldots x_n), L_{m_1,m_2,\ldots,m_n}(\ -g\,;x_1,\ldots x_n)\} \tag{2.12}$$

for any $g \in C_b(I^n)$.

Applying this result to $G(s_1,\ldots s_n) = \Delta_{s_1,\ldots,s_n}[f;x_1,\ldots,x_2]$ and using the monotonicity of $L_{m_1,m_2,\ldots,m_n}$ and the lemma 2.1, we find (with $A(\varepsilon) = \max\{A_1(\varepsilon),\ldots,A_n(\varepsilon)\}$) the inequality:

$$|f\ (x_1,\ldots x_n) - U_{m_1,m_2,\ldots,m_n}(f;x_1,\ldots x_n)| \leq$$

$$\leq L_{m_1,m_2,\ldots,m_n}\left[\frac{\varepsilon}{n+1} + A(\varepsilon)\sum_{i=1}^{n}(x_i - \cdot)^2\,;x_1,\ldots,x_n\right]. \tag{2.13}$$

After some transformation of (2.13) we obtain:

$$|f\ (x_1,\ldots x_n) - U_{m_1,m_2,\ldots,m_n}(f;x_1,\ldots x_n)| \leq$$
$$\leq \frac{\varepsilon}{n+1} + A(\varepsilon)\,L_{m_1,m_2,\ldots,m_n}\left(\sum_{i=1}^{n} e_i^2;x_1,\ldots,x_n\right) -$$
$$-2\cdot A(\varepsilon)\sum_{i=1}^{n} x_i \cdot L_{m_1,m_2,\ldots,m_n}(e_i;x_1,\ldots,x_n) +$$

$$+A(\varepsilon)\cdot L_{m_1,m_2,\dots,m_n}(e_0;x_1,\dots,x_n)\sum_{i=1}^{n}x_i^2. \tag{2.14}$$

Using now the hypothesis of the theorem, we can write:

$$|f\ (x_1,\dots x_n)-U_{m_1,m_2,\dots,m_n}(f;x_1,\dots x_n)|\le$$

$$\le\frac{\varepsilon}{n+1}+A(\varepsilon)\cdot\left\{\delta_{m_1,\dots,m_n}(x_1,\dots,x_n)-2\cdot\sum_{i=1}^{n}x_i\alpha^{(i)}_{m_1,\dots,m_n}(x_1,\dots,x_n)\right\}. \tag{2.15}$$

Taking into account that $\left\{\alpha^{(i)}_{m_1,\dots,m_n}(x_1,\dots,x_n)\right\},\{\delta_{m_1,\dots,m_n}(x_1,\dots,x_n)\}$ tend to zero uniformly on I^n as $m_1,m_2,\dots,m_n$ tend to infinity, from (2.15) we obtain the desired result.

Remark 2.1: The positive operator $L_{m_1,m_2,\dots,m_n}:C_b(I^n)\to C_b(I^n)$ is the product of the parametric extension $L^{x_1}_{m_1},\dots,L^{x_n}_{m_n}$ of the positive linear univariate operator $L_m:\mathbf{R}^I\to\mathbf{R}^I$.

Remark 2.2: In the case n=2, the theorem 2.1 reduced to the Korovkin-type theorem established in [1]. The idea of the proof of the theorem 2.1 is suggested by the idea from [1].

3. We shall present two applications of the theorem 2.1. For simplicity, we consider the case n=3.

Example 1: We consider the Bernstein-Stancu´s operator $B^{\langle\alpha\rangle}_{m_1},B^{\langle\beta\rangle}_{m_2},B^{\langle\gamma\rangle}_{m_3}:\mathbf{R}^I\to\mathbf{R}^I$, given by

$$B^{\langle\alpha\rangle}_{m_1}(f)(x)=\sum_{i=1}^{m_1}f\left(\frac{i}{m_1}\right)\cdot\omega_{m_1,i}(x,\alpha),x\in I,$$

$$B^{\langle\beta\rangle}_{m_2}(g)(y)=\sum_{j=1}^{m_2}f\left(\frac{j}{m_2}\right)\cdot\omega_{m_2,j}(y,\beta),y\in I,$$

$$B^{\langle\gamma\rangle}_{m_3}(h)(z)=\sum_{k=1}^{m_3}f\left(\frac{k}{m_3}\right)\cdot\omega_{m_3,k}(z,\gamma),z\in I,$$

where $\omega_{m_1,i}(x,\alpha)$,$\omega_{m_2,j}(y,\beta),\omega_{m_3,k}(z,\gamma)$ are the fundamental polynomials of Bernstein-Stancu type, i.e.

$$\omega_{m_1,i}(x,\alpha)=\begin{pmatrix}m_1\\ i\end{pmatrix}\frac{x^{[i,-\alpha]}\cdot(1-x)^{[m_1-i,-\alpha]}}{1^{[m_1,-\alpha]}},$$

$$\omega_{m_2,j}(y,\beta)=\begin{pmatrix}m_2\\ j\end{pmatrix}\frac{y^{[j,-\beta]}\cdot(1-y)^{[m_2-j,-\beta]}}{1^{[m_2,-\beta]}},$$

$\omega_{m_3,k}(z,\gamma) = \begin{pmatrix} m_3 \\ k \end{pmatrix} \frac{z^{[k,-\gamma]}\cdot(1-z)^{[m_3-k,-\gamma]}}{1^{[m_3,-\gamma]}}.$

In the precedent relation, $x^{[i,-\alpha]}$ denotes the factorial power of x with the exponent i and the increment -α, i.e.$x^{[i,-\alpha]} = x(x+\alpha)\dots(x+(i-1)\alpha)$.

In same relation, the parameter α, β and γ satisfy the condition $\alpha = \alpha(m_1) \geq 0$, $\beta = \beta(m_2) \geq 0$, $\gamma = \gamma(m_3) \geq 0$.

Let suppose that $f \in C_b(I^3)$; the operators $L_{m_1}, L_{m_2}, L_{m_3} : C_b(I^3) \to C_b(I^3)$ are the parametric extensions of the operator $B_{m_1}^{\langle\alpha\rangle}, B_{m_2}^{\langle\beta\rangle}, B_{m_3}^{\langle\gamma\rangle}$:

$L_{m_1}(f)(x,y,z) = \sum_{i=1}^{m_1} f\left(\frac{i}{m_1}, y, z\right)\cdot\omega_{m_1,i}(x,\alpha), x \in I;$

$L_{m_2}(f)(x,y,z) = \sum_{i=1}^{m_1} f\left(x, \frac{j}{m_2}, z\right)\cdot\omega_{m_2,j}(y,\beta), y \in I;$

$L_{m_3}(f)(x,y,z) = \sum_{k=1}^{m_3} f\left(x, y, \frac{k}{m_3}\right)\cdot\omega_{m_3,k}(z,\gamma), z \in I.$

The operator L_{m_1,m_2,m_3} is the product of the operators $L_{m_1}, L_{m_2}, L_{m_3}$ and it is defined by

$$L_{m_1,m_2,m_3}(f)(x,y,z) = \\ = \sum_{i=0}^{m_1}\sum_{i=0}^{m_1}\sum_{k=0}^{m_3} f\left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3}\right)\cdot\omega_{m_1,i}(x,\alpha)\cdot\omega_{m_2,j}(y,\beta)\cdot\omega_{m_3,k}(z,\gamma).$$

By direct computation, one obtains

$L_{m_1,m_2,m_3}(e_0)(x,y,z) = 1, L_{m_1,m_2,m_3}(e_1)(x,y,z) = x,$

$L_{m_1,m_2,m_3}(e_2)(x,y,z) = y, L_{m_1,m_2,m_3}(e_3)(x,y,z) = z,$

$L_{m_1,m_2,m_3}\left(e_1^2+e_2^2+e_3^2\right)(x,y,z) = x^2+y^2+z^2+$
$+\frac{x(1-x)}{1+\alpha}\left[\frac{1}{m_1}+\alpha\right] + \frac{y(1-y)}{1+\beta}\left[\frac{1}{m_2}+\beta\right] + \frac{z(1-z)}{1+\gamma}\left[\frac{1}{m_3}+\gamma\right]$

for any $(x,y,z) \in I^3$. It follows that the sequence $\{L_{m_1,m_2,m_3}\}_{m_1,m_2,m_3\in\mathbf{N}}$ satisfies the hypothesis of the theorem 2.1 with

$$\alpha_{m_1,m_2,m_3}^{(1)} = \alpha_{m_1,m_2,m_3}^{(2)} = \alpha_{m_1,m_2,m_3}^{(3)} = 0$$

and

$$\delta_{m_1,m_2,m_3}(x_1,x_2,x_3) = \frac{x(1-x)}{1+\alpha}\left[\frac{1}{m_1}+\alpha\right]+ \\ +\frac{y(1-y)}{1+\beta}\left[\frac{1}{m_2}+\beta\right] + \frac{z(1-z)}{1+\gamma}\left[\frac{1}{m_3}+\gamma\right]$$

If $\alpha(m_1), \beta(m_2), \gamma(m_3)$ tend to zero as m_1, m_2, m_3 tend to infinity, from the theorem 2.1 one obtains that the sequence $\{U_{m_1,m_2,m_3}(f)\}$, defined by

$$U_{m_1,m_2,m_3}(f)(x,y,z) =$$

$$= \sum_{i=0}^{m_1} \sum_{i=0}^{m_1} \sum_{k=0}^{m_3} f\left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3}\right) \cdot \omega_{m_1,i}(x,\alpha) \cdot \omega_{m_2,j}(y,\beta) \cdot \omega_{m_3,k}(z,\gamma) \cdot$$

$$\cdot \left\{ f\left(\frac{i}{m_1}, y, z\right) + f\left(x, \frac{j}{m_2}, z\right) + f\left(x, y, \frac{k}{m_3}\right) - \right.$$

$$\left. -f\left(\frac{i}{m_1}, \frac{j}{m_2}, z\right) - f\left(\frac{i}{m_1}, y, \frac{k}{m_3}\right) - f\left(x, \frac{j}{m_2}, \frac{k}{m_3}\right) + f\left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3}\right) \right\}$$

converges to f, uniformly on I^3, as m_1, m_2, m_3 tend to infinity, for any $f \in C_b\left(I^3\right)$.

This result was obtained first in the paper [1.3] without the theorem 2.1.

Example 2: In this example one consider the operator of Bleimann, Butzer and Hahn $\widetilde{L}, \overline{L}, \overline{\overline{L}} : \mathbf{R}^I \to \mathbf{R}^I$, given by

$$\widetilde{L}(f)(x) = \sum_{i=0}^{m_1} f\left(\frac{i}{m_1 - i + 1}\right) \cdot p_{m_1,i}(x), \qquad p_{m_1,i}(x) = \begin{pmatrix} m_1 \\ i \end{pmatrix} \cdot \frac{x^i}{(1+x)^{m_1}};$$

$$\overline{L}(g)(y) = \sum_{j=0}^{m_2} g\left(\frac{j}{m_2 - j + 1}\right) \cdot \overline{q}_{m_2,j}(y), \qquad \overline{q}_{m_2,j}(y) = \begin{pmatrix} m_2 \\ j \end{pmatrix} \cdot \frac{y^j}{(1+y)^{m_2}};$$

$$\overline{\overline{L}}(h)(z) = \sum_{k=0}^{m_3} h\left(\frac{k}{m_3 - k + 1}\right) \cdot \overline{\overline{r}}_{m_3,k}(z), \qquad \overline{\overline{r}}_{m_3,k}(z) = \begin{pmatrix} m_3 \\ k \end{pmatrix} \cdot \frac{z^{jk}}{(1+z)^{m_3}}.$$

The operator $L_{m_1}, L_{m_2}, L_{m_3}$ are the parametric extensions of the operators from above, i.e.

$$L_{m_1}(f)(x,y,z) = \sum_{i=0}^{m_1} f\left(\frac{i}{m_1 - i + 1}, y, z\right) \cdot p_{m_1,i}(x),$$

$$L_{m_2}(f)(x,y,z) = \sum_{j=0}^{m_2} g\left(x, \frac{j}{m_2 - j + 1}, z\right) \cdot \overline{q}_{m_2,j}(y),$$

$$L_{m_3}(f)(x,y,z) = \sum_{k=0}^{m_3} h\left(x, y, \frac{k}{m_3 - k + 1}\right) \cdot \overline{\overline{r}}_{m_3,k}(z).$$

The product of these extensions is the positive linear operator

$$L_{m_1,m_2,m_3}(f)(x,y,z) = \sum_{i=0}^{m_1} \sum_{i=0}^{m_1} \sum_{k=0}^{m_3} f\left(\frac{i}{m_1}, \frac{j}{m_2}, \frac{k}{m_3}\right) \cdot p_{m_1,i}(x) \cdot \overline{q}_{m_2,j}(y) \cdot \overline{\overline{r}}_{m_3,k}(z).$$

It is easy to see that $\{L_{m_1,m_2,m_3}\}$ satisfies the hypothesis of theorem 2.1. Applying then this theorem, it follows that the sequence $\{U_{m_1,m_2,m_3}(f)\}$, where

$$U_{m_1,m_2,m_3}(f)=\sum_{i=0}^{m_1}\sum_{i=0}^{m_1}\sum_{k=0}^{m_3} f\left(\frac{i}{m_1},\frac{j}{m_2},\frac{k}{m_3}\right)\cdot p_{m_1,i}(x)\cdot \overline{q}_{m_2,j}(y)\cdot \overline{\overline{r}}_{m_3,k}(z)\cdot$$

$$\cdot\qquad \cdot\left\{f\left(\frac{i}{m_1},y,z\right)+f\left(x,\frac{j}{m_2},z\right)+f\left(x,y,\frac{k}{m_3}\right)-\right.$$

$$\left.-f\left(\frac{i}{m_1},\frac{j}{m_2},z\right)-f\left(\frac{i}{m_1},y,\frac{k}{m_3}\right)-f\left(x,\frac{j}{m_2},\frac{k}{m_3}\right)+f\left(\frac{i}{m_1},\frac{j}{m_2},\frac{k}{m_3}\right)\right\}$$

converges to f, uniformly on I^3 as m_1, m_2, m_3 tend to infinity.

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